ME 550. FOUNDATIONS OF ENGINEERING SYSTEMS ANALYSIS

Appendix #04: INTRODUCTION TO TOPOLOGY AND MEASURE THEORY

The notion of topology allows generalization of open sets and continuity of functions beyond metric spaces described in Chapter 02. Topology is a vast subject and therefore only the rudimentary concepts of topology are presented in this section. We also present rudimentary concepts of Measure Theory in this section. The objective is to bring in these concepts to those who have never been exposed to Measure Theory. Examples of easily readable books on these topics are Real Analysis by Royden (1989) and Elements of Integration and Lebesgue Measure by Bartle (1966). Also refer to Appendix D of Naylor and Sell (1982).

Topology and Topological Spaces

Definition A4-1: Let Ω be a nonempty set and let $\Im$ be a collection of subsets of Ω such that:

- $\varnothing \in \Im$ and $\Omega \in \Im$.
- If $S_k \in \Im$ for $k = 1,2,\cdots,n$, then $\bigcap_{k=1}^{n} S_k \in \Im$ — finite intersection
- If $S_\alpha \in \Im$ for $\alpha \in I$ where I is the index set, then $\bigcup_{\alpha \in I} S_\alpha \in \Im$ — arbitrary union

Then, $\Im$ is a topology of Ω; $\langle \Omega, \Im \rangle$ is a topological space; and each member of $\Im$ is said to be a $\Im$-open set in Ω. ♦

Definition A4-2: The usual topology $\langle \Re, \mathsf{U} \rangle$ is defined with $\Omega = \Re \equiv (-\infty, \infty)$ and $\Im = \mathsf{U}$ that contains all open intervals in $\Re$. A set $G \subseteq \Re$ is said to be U-open (i.e., open relative to the usual topology U) if either $G = \varnothing$ or, for $G \neq \varnothing$, $\forall p \in G$ $\exists$ an open interval $(a,b) \subset G$ such that $p \in (a,b)$. ♦

Definition A4-3: Let $\langle X, \Im \rangle$ be a topological space. Then the complement of every $\Im$-open set in X is said to be $\Im$-closed in X. That is, if $S \in \Im$, then $S^c \equiv X - S$ is $\Im$-closed in X. In other words, S is $\Im$-open in X if and only if $S^c \equiv X - S$ is $\Im$-closed in X. ♦

Definition A4-4: Let $\langle X, \Im \rangle$ be a topological space and let $p \in X$. Then, $B \subseteq X$ is called a $\Im$-neighborhood of $p \in X$ if $\exists$ a $\Im$-open set G such that $p \in G \subseteq B$.

Remark A4-1: Note that, in the topological sense, a $\Im$-neighborhood of a point $p \in X$ need not be a $\Im$-open set in X. However, a $\Im$-open set is a $\Im$-neighborhood of each of its points. ♦

Definition A4-5: Let $S \subset X$ where $\langle X, \Im \rangle$ is a topological space. Then, a point $p \in X$ is said to be a cluster point of S if every $\Im$-neighborhood of p contains at least one point of S other than p. In other words, p is a cluster point of S if and only if the following condition holds: B is a $\Im$-neighborhood of p implying that $(B - \{p\}) \cap S \neq \varnothing$.

Example A4-1: Consider the open interval $(0,1) \subset \Re$. In the usual topology $\langle \Re, \mathsf{U} \rangle$, both 0 and 1 are cluster points of (0,1). Furthermore, every point of (0,1) is a cluster point of (0,1). ♦

Definition A4-6: Let $\langle X, \Im \rangle$ and $\langle Y, \vartheta \rangle$ be two topological spaces. Then a mapping $f : X \to Y$ is said to continuous (more precisely, $\Im - \vartheta$ continuous) if the inverse image $f^{-1}(\Phi)$ is $\Im$-open in X for every ϑ-open set Φ in Y. ♦

Definition A4-7: A $\Im-\vartheta$ continuous mapping $f:X\rightarrow Y$ is $\Im-\vartheta$-bicontinuous if $f[\Theta]$ is a $\vartheta-$open set in Y for every $\Im$-open set Θ in X. ♦

Definition A4-8: bijective and $\Im-\vartheta$-bicontinuous mapping $f:X\rightarrow Y$ is called a $\Im-\vartheta$-homeomorphism of X and Y. ♦

Remark A4-2: Two topological spaces are equivalent if they are homeomorphic. ♦

Definition A4-9: A topological space $\langle X,\Im\rangle$ is called a Hausdorff space (or a T_2-space) if, for every pair of distinct points x and y, i.e., $x,y\in X$ and $x\neq y$, $\exists$ $\Im$-neighborhoods B_x and B_y such that $B_x\cap B_y=\varnothing$. ♦

Example A4-2: The usual topology $\langle\Re,\mathsf{U}\rangle$ where the collection of all open subsets (defined in the usual sense) of $\Re=(-\infty,\infty)$ is Hausdorff. ♦

Example A4-3: Let $X=\{a,b,c\}\subset\Re$ and $\Im=\{\varnothing,\{a,b\},\{c\},X\}$. Clearly, the topological space is not Hausdorff because a and b are distinct points of X that do not have disjoint $\Im$-neighborhoods.

Definition A4-10: Let $\langle X,\Im\rangle$ be a topological space and $Y\subseteq X$. The $\Im$-relative topology of Y, denoted as $\Im_Y$, is defined as: $\Im_Y=\{G\cap Y:G\in\Im\}$. Then, $\langle Y,\Im_Y\rangle$ is called a subspace of $\langle X,\Im\rangle$. ♦

HW A4-1: Show that $\Im_Y$ is a topology of Y. ♦

Example A4-4: Let $X=\{a,b,c\}\subset\Re$ and $\Im=\{\varnothing,\{a,b\},\{c\},X\}$. Clearly, the topological space $\langle X,\Im\rangle$ is not Hausdorff because a and b are distinct points of X that do not have disjoint $\Im$-neighborhoods. ♦

Example A4-5: Let $Y=(0,1)\subset\Re$. Consider the relative topology $\langle Y,U_Y\rangle$ in which the interval $(\frac{1}{2},1)$ is open in $\langle Y,U_Y\rangle$. Although $[\frac{1}{2},1)$ is not closed in $\langle\Re,\mathsf{U}\rangle$ but $[\frac{1}{2},1)$ is closed in $\langle Y,U_Y\rangle$ because $[\frac{1}{2},1)$ is the complement of the U_Y-open set $(0,\frac{1}{2})$ in Y. The set $\left\{\frac{1}{k}:k\in\mathbf{N}\right\}$ is closed in the relative topology $\langle Y,U_Y\rangle$, $\langle\Re,\mathsf{U}\rangle$ because it has no cluster points in Y. However, $\left\{\frac{1}{\mathrm{k}}:\mathrm{k}\in\mathbf{N}\right\}$ is not closed in the usual topology $\langle\Re,\mathsf{U}\rangle$ because the cluster point 0 is not contained in $\left\{\frac{1}{k}:k\in\mathbf{N}\right\}$. ♦

Next, we present three important results:

Result A4-1: The topological spaces $\langle\Re,\mathsf{U}\rangle$ and $\langle(0,1),\mathsf{U}_{(0,1)}\rangle$ are homeomorphic. This result follows by constructing a bijective and bicontinuous function $f:(0,1)\rightarrow\Re$ such as $f(x)=\frac{2x-1}{x(x-1)}$. ♦

Result A4-2: If I_1 and I_2 are two U-open intervals in $\Re$, the spaces $\langle \mathrm{I}_1,\mathsf{U}_{\mathrm{I}1}\rangle$ and $\langle I_1,\mathsf{U}_{I2}\rangle$ are homeomorphic. ♦

Result A4-3: If I_1 and I_2 are two U-closed intervals in $\Re$, the spaces $\langle I_1,\mathsf{U}_{I1}\rangle$ and $\langle I_1,\mathsf{U}_{I2}\rangle$ are homeomorphic. ♦

Compactness in a Topological Space

Definition A4-11: A metric space $\langle S,d\rangle$ is sequentially compact if every sequence in S has a convergent subsequence. ♦

Example A4-6: The sequence $\{1,\frac{1}{2},3,\frac{1}{4},5,\frac{1}{6},\cdots\}$ has a convergent subsequence $\left\{\frac{1}{2^k} : k \in \mathbf{N}\right\}$ in $\Re$. Note that the sequence $\{1,\frac{1}{2},3,\frac{1}{4},5,\frac{1}{6},\cdots\}$ itself is not convergent in $\Re$ and that it contains many subsequences like $\{1,3,5,7,\cdots\}$ which are not convergent. ♦

Example A4-7: The set $(0,1]$ is not compact in $\langle \Re, \mathsf{U} \rangle$ because the sequence $\left\{\frac{1}{k} : k \in \mathbf{N}\right\}$ does not have a subsequence with a limit point in $(0,1]$. ♦

Definition A4-12: Let $\langle S, \Im \rangle$ be a topological space and let $E \subseteq S$. Let $\Sigma = \{S_\alpha : \alpha \in I\}$ be a collection of subsets of S where I is an index set (which nonempty, and finite or countable or uncountable). Then, Σ is said to be a covering of E if $E \subseteq \bigcup_{\alpha \in I} S_\alpha$. If Σ_1 is a covering of E and Σ_2 is a covering of E such that $\Sigma_2 \subseteq \Sigma_1$, then Σ_2 is a subcovering of Σ_1. ♦

HW A4-2: Let $\Sigma_1 = \left\{(0,\frac{k}{k+1}) : k \in \mathbf{N}\right\}$. Verify that Σ_1 is a U-open covering of (0,1) ♦

Example A4-8: If $\Sigma_2 = \left\{\left(0,\frac{4k+3}{4(k+1)}\right) : k \in \mathbf{N}\right\}$ is a subcovering of Σ_1, then $\Sigma_1 = \left\{(0,\frac{k}{k+1}) : k \in \mathbf{N}\right\}$ ♦

Definition A4-13: Let $\langle S, \Im \rangle$ be a topological space. A covering Σ of $E \subseteq S$ is said to be a $\Im$-open covering of E if every member of Σ is a $\Im$-open set. A covering Σ of a set E is said to be finite if card(Σ) is finite. ♦

Definition A4-14: A topological space $\langle S, \Im \rangle$ is said to be compact if every $\Im$-open covering of S has a finite subcovering. ♦

Theorem A4-1: The topological space $\langle \Re, \mathsf{U} \rangle$ is not compact. Therefore, no open interval on $\Re$ is compact in its relativized U-topology.

Proof of_Theorem A4-1: Let $\Sigma = \{(-k,k) : k \in \mathbf{N}\}$. Then, Σ is a U-open covering of $\Re$ because each member of Σ is an open interval in $\Re$, and $\Re \subset \bigcup_{k \in \mathbf{N}} (-k,k)$. If $x \in \Re$, then $\exists\, n_x \in \mathbf{N}$ such that $n_x > |x| \Rightarrow x \in (-n_x, n_x)$. So, $x \in \bigcup_{n \in \mathbf{N}} (-n,n)$. Now, let $(-n_1,n_1),(-n_2,n_2),\cdots,(-n_k,n_k)$ be any finite collection of members of Σ and let $n^* = \max(n_1,n_1,\cdots,n_k)$. Then, $n^* \notin \bigcup_{k \in \mathbf{N}} (-n_k, n_k)$. Therefore, there is no finite collection of members of Σ which is a covering of $\Re$. So, $\langle \Re, \mathsf{U} \rangle$ is not compact. The second assertion follows from homeomorphism between $\langle \Re, \mathsf{U} \rangle$ and $\langle \mathbf{I}, \mathsf{U} \rangle$ where $\mathbf{I}$ is an interval in $\Re$. ♦

Definition A4-15: A mapping $f : X \to \Re$ is bounded if the range of $f[X]$ is a bounded subset of $\Re$. ♦

Next we present the following important results without proof.

Result A4-4: If $\langle X, \Im_X \rangle$ is a compact subspace of a Hausdorff space $\langle \Omega, \Im \rangle$, then X is $\Im$-closed. ♦

Result A4-5: If $\langle \Omega, \Im \rangle$ is compact and X is a $\Im$-closed subset of X , then $\langle X, \Im_X \rangle$ is compact. ♦

Result A4-6: (Heine-Borel Theorem) For $X \subset \Re$, $\langle X, \mathbf{U}_X \rangle$ is compact iff X is bounded and $\mathbf{U}$-closed. ♦

Result A4-7: A continuous image of a compact space is compact. That is, for two topological spaces $\langle X, \Im \rangle$ and $\langle Y, \vartheta \rangle$, if $f : X \to Y$ is $\Im - \vartheta$-continuous, compactness of $\langle X, \Im \rangle$ implies compactness of $\langle Y, \vartheta \rangle$. ♦

Result A4-8: Let $\langle X, \Im \rangle$ be a compact space and let $\langle Y, \vartheta \rangle$ be a Hausdorff space. If $f : X \to Y$ is $\Im - \vartheta$-continuous and surjective, then f is a homeomorphism. ♦

Result A4-9: Let $\langle X, \Im \rangle$ be a compact space. If $f : X \to \Re$ is $\Im - \mathsf{U}$-continuous, then $f(\cdot)$ is bounded. ♦

Result A4-10: (Bolzano-Weierstrass Theorem): Every bounded infinite subset of $\Re$ has at least one U-cluster point. ♦

Result A4-11: Let $X \subseteq \Re$ be bounded and U-closed. If $f : X \to \Re$ is $\mathsf{U}_X - \mathsf{U}$-continuous, then f is bounded. ♦

Total Boundedness and Approximation

Definition A4-16: Let E be a set in a metric space $\langle S, d \rangle$. Given $\varepsilon > 0$, $E_\varepsilon \subset E$ is an $\varepsilon-$net of E if :

(i) E_ε is a finite set; and (ii) $\forall x \in E, \exists y \in E_\varepsilon$ such that $d(x, y) < \varepsilon.$ ♦

Definition A4-17: A set E in a metric space $\langle S, d \rangle$ is totally bounded if: $\forall \varepsilon > 0, \exists$ an $\varepsilon-$net in E. ♦

Remark A4-3: Total boundedness implies boundedness. The converse is true for all finite-dimensional spaces but, in general, it is not true for infinite-dimensional spaces. ♦

Remark A4-3: Every finite set in a metric space is bounded and hence it is totally bounded. ♦

Example A4-8: Consider the closed ball $\tilde{B}_1(\underline{0}) = \{x \in \ell_2 : \|x\|_{\ell_2} \le 1\}$ where the distance function is defined as:

$$d(x, y) = \|x - y\|_{\ell_2} \equiv \sqrt{\sum_{k=1}^{\infty} |x_k - y_k|^2} \ \forall x, y \in \ell_2$$

The set $\tilde{B}_1(\underline{0})$ is bounded because $d(x, y) \le 2 \ \forall x, y$ but $\tilde{B}_1(\underline{0})$ is not totally bounded as seen below.

Let us construct $E = \{e^k : k \in \mathsf{N}\}$ where e^k is the sequence of all 0's except '1' as the k^{th} element of the sequence. Clearly, $d(e^k, e^\ell) = \sqrt{2}\delta_\ell^k$. If an $\varepsilon-$net $E_{1/2}$ exists for $\varepsilon = \frac{1}{2}$, then $E_{1/2}$ must be a finite subset in E. But since the closed balls $\tilde{B}_{1/2}(e^k)$ and $\tilde{B}_{1/2}(e^\ell)$ are disjoint for all $k \ne \ell$, $E_{1/2}$ must contains a point within a distance $\frac{1}{2}$ of each e^k. Since there are countably many e^k's, $E_{1/2}$ cannot be a finite set. However, notice that this violation of finite cardinality would not have occurred in a finite-dimensional space.

Next we present the following important results without proof.

Result A4-12: Let $E \subseteq X$ in a metric space $\langle X, d \rangle$. If E has an $\varepsilon-$net for some $\varepsilon > 0$, then E is bounded. ♦

Result A4-13: Let $\langle X, d \rangle$ be a totally bounded metric space. Then, X is separable. ♦

Result A4-14: A bounded set $E \subseteq \ell_2$ is totally bounded iff $\forall \varepsilon > 0 \ \exists \ n(\varepsilon) \in \mathsf{N}$ such that

$$\sum_{k=n}^{\infty} |x_k|^2 < \varepsilon \ \forall x \in E$$ ♦

Result A4-15: Let $E \subseteq X$ in a metric space $\langle X, d \rangle$. Then, the following statements are equivalent:

(i) The closure $\overline{E}$ is sequentially compact.

(ii) Every sequence in E has a subsequence that converges in X. ♦

Result A4-16: Every sequentially compact set in a metric space is closed. ♦

Result A4-17: Every sequentially compact metric space is complete. ♦

Result A4-18: Every sequentially compact metric space is totally bounded. ♦

Result A4-19: A metric space is sequentially compact iff it is totally bounded and complete. ♦

Concept of Measure and Measurable Spaces

Intuitively, Lebesgue measure is the length of an interval or of an at most countable union of intervals on the real line $\Re$. This concept can be extended to $\Re^2$ and $\Re^3$ as areas and volumes, and also to other finite-dimensional spaces. The concept of axiomatic probability theory, introduced by Kolmogorov, is based on the principle of Lebesgue measure. In general, measure is a set function, i.e., an assignment of a number m(A) to each set in a certain class.

Consider the open interval (a,b) whose measure is the length $(b-a)$. Similarly, the length of a countable union of disjoint open intervals can be obtained by summing the lengths of these intervals. Defining sets only in terms of disjoint open intervals is often restrictive. Therefore, we would like to generalize the concept of measure. At this stage, solely for simplicity, we would restrict the treatment of measure to finite intervals and bounded subsets of $\Re$.

- For a finite interval $\mathbf{I}$ (open, closed or semi-open), the measure is equal to the length of the interval, i.e., $m(\mathbf{I}) = \ell(\mathbf{I})$.
- If $\{\mathbf{I_k}\}$ is an at most countable (i.e., finite or countably infinite) sequence of disjoint intervals, then $m\left(\bigcup_{i=1}^{\infty}\mathbf{I}_i\right) = \sum_{i=1}^{\infty}\ell(\mathbf{I}_i)$.
- The measure is translation-invariant. That is, if E is a set for which the measure m is defined and if, for any given $y \in \Re$, $E \oplus y$ is the set $\{x+y : x \in E\}$ obtained by replacing each point x in E by $x+y$, then $m(E \oplus y) = m(E)$.

<u>Outer Measure</u>: For each set $E \subseteq \Re$, consider the countable collection $S = \{\mathbf{I}_k\}$ of open intervals in $\Re$ that cover E, i.e., $E \subseteq \bigcup_{i=1}^{\infty}\mathbf{I}_i$. The outer measure of E is then defined as:

$$\overline{m}(E) = \inf_{all S}\sum_{j=1}^{\infty}\ell(\mathbf{I}_j)$$

Clearly, if $A \subseteq B$, then $\overline{m}(A) \le \overline{m}(B)$. Also, $\overline{m}(\varnothing) = 0$ and each set consisting of a single point has a zero outer measure. The following results are presented from Royden (1989) without proof:

- <u>Result 1</u>: The outer measure of an interval (open or closed or semi-open) is its length.
- <u>Result 2</u>: If $\{A_k\}$ is a sequence of subsets of $\Re$, then $\overline{m}\left(\bigcup_{k=1}^{\infty}A_k\right) \le \sum_{k=1}^{\infty}\overline{m}(A_k)$.
- <u>Result 3:</u> If A is a countable subset (dense or not) of $\Re$, then $\overline{m}(A) = 0$.
- <u>Result 4</u>: If A is an uncountable set, then $\overline{m}(A) \ge 0$. Usually the measure of an uncountable set is greater than 0. However, there are uncountable sets such as the Cantor set whose measure is 0.

Definition A4-18: A set $E \subseteq \Re$ is said to be measurable if, for every $A \subseteq \Re$, the following condition holds: $\overline{m}(A) = \overline{m}(A \cap E) + \overline{m}(A \cap E^c)$. ♦

The fact that $A = (A \cap E) \cup (A \cap E^c)$ implies, by virtue of Result 2, that $\overline{m}(A) \le \overline{m}(A \cap E) + \overline{m}(A \cap E^c)$. Therefore, E is measurable whenever $\overline{m}(A) \ge \overline{m}(A \cap E) + \overline{m}(A \cap E^c)$. Furthermore, because of symmetry, E^c is measurable if and only if E is measurable.

<u>Inner Measure</u>: Assuming that E is a bounded set, we choose an interval $\mathbf{I}^*$ such that $E \subset \mathbf{I}^*$. We define the inner measure $\underline{m}$ as: $\underline{m}(E) = \ell(\mathbf{I}^*) - \overline{m}(\mathbf{I}^* - E)$. The following results are presented without proof:

- <u>Result 5</u>: The inner measure $\underline{m}(E)$ is invariant for every $\mathbf{I}^*$ containing E.

- Result 6: For every bounded set E, then $\underline{m}(E) \le \overline{m}(E) < \infty$.
- Result 7: If E is a finite interval, then $\underline{m}(E) = \overline{m}(E) = \ell(E)$.
- Result 8: A bounded set $E \subseteq \Re$ is measurable if $\underline{m}(E) = \overline{m}(E)$. In that case, we denote the measure as: $m(E) = \underline{m}(E) = \overline{m}(E)$.

Demonstration of the Existence of a Nonmeasurable Set: The usually encountered sets are measurable and, for engineering applications, we may not have to deal with any nonmeasurable sets. Indeed, it is not easy to find a nonmeasurable set. However, from the conceptual point of view, it is important to establish the existence of a non-measurable set. An example of a nonmeasurable set [Royden (1989), pp. 64-65] is given below.

Let $x, y \in [0,1)$. Define the sum Modulo 1 as: $x \oplus y = \begin{cases} x+y & \text{if } (x+y) < 1 \\ x+y-1 & \text{if } (x+y) \ge 1 \end{cases}$

where the operator $\square$ can be interpreted as follows: If $\theta = x \oplus y$, then the angle $2\pi\theta$ in radians is the sum modulo addition of two angles $2\pi x$ and $2\pi y$. The sum modulo operator $\oplus$ can also be translated, i.e., $E \oplus y = \{z : z = x \oplus y \text{ for } x \in E\} \quad \forall y \in E$. Furthermore, the operator $\oplus$ is commutative and associative. We establish the following lemma before citing an example of a nonmeasurable set.

Lemma: Let $E \subseteq [0,1)$ be a measurable set. Then, the set $E \oplus y$ is measurable and m($E \oplus y$) = m(E) $\forall y \in E$.

Proof: Let $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$ for some $y \in [0,1)$.. Therefore, $E_1 \cap E_2 = \varnothing$ and $E_1 \cup E_2 = E$ which imply that $m(E_1) + m(E_2) = m(E)$. Since the set $E_1 + y$ is measurable and $E_1 \oplus y = E_1 + y$, we argue that $E_1 \oplus y$ is also measurable and m($E_1 \oplus y$) = m($E_1 + y$) = m(E_1). Similarly, $E_2 \oplus y = E_2 + y - 1$ implies that $E_2 \oplus y$ is measurable and m($E_2 \oplus y$) = m(E_2). But $(E_1 \oplus y) \cup (E_2 \oplus y) = E \oplus y$ and $(E_1 \oplus y) \cap (E_2 \oplus y) = \varnothing$. Hence,

$$m((E \oplus y)) = m(E_1 \oplus y) + m(E_2 \oplus y) = m(E_1) + m(E_2) = m(E). \qquad \blacklozenge$$

Now we proceed to construct a nonmeasurable set. Let $x, y \in [0,1)$ such that $(x - y)$ is rational. This is an equivalence relation $x \sim y$ because: (i) $x \sim x$ since 0 is rational; (ii) $x \sim y \equiv y \sim x$ since if r is rational, so is $-r$; and (iii) ($x \sim y$ and $y \sim z$) $\Leftrightarrow x \sim z$ because if r and $\tilde{r}$ r' are rational, so is $r + \tilde{r}$. Next, we partition the set $[0,1)$ into equivalence classes such that any two elements of a given equivalence class differ by a rational, and any two numbers belonging to different equivalence classes differ by an irrational.

Let us construct the set P that contains exactly one number from each equivalence class and assume that P is a measurable set. Let $r_i,\ i = 0,1,2,\cdots,$ be an enumeration of the rational numbers in $[0,1)$ with $r_0 = 0$. Let us define $P_i = P \oplus r_i \Rightarrow P_0 = P$ and let $x \in P_i \cap P_j$ for $i \ne j$. Therefore, $x = q_i \oplus r_i = q_j \oplus r_j$ with $q_i, q_j \in P$. Then, $q_i - q_j = r_i - r_j$ is a rational and hence $q_i \sim q_j$. Since P has exactly one element from each equivalence class, $q_i = q_j$. That means, for $i \ne j$, $P_i \cap P_j = \varnothing$. Therefore, the collection of sets $\{P_i\}$ is pairwise disjoint. On the other hand, each $x \in [0,1)$ belongs to one and only one of the equivalence classes and therefore must be equivalent to an element of P. But, if x differs from an element of P by a rational r_i, then $x \in P_i$ for some $i \in \{1,2,3,\cdots\}$. Therefore, $\bigcup_{i=1}^{\infty} P_i = [0,1)$. Further, since P_i is a translation modulo 1 of P, we conclude by the lemma, that each P_i is measurable and has the same measure as P. If it is so, $m([0,1)) = \sum_{i=0}^{\infty} m(P_i) = \sum_{i=0}^{\infty} m(P)$ which implies that $m([0,1))$ is either 0 or ∞ depending on whether $m(P)$ is zero or non-zero. But we know that $m([0,1)) = 1$. This is a contradiction. So P is a nonmeasurable set. ♦

Measurable Functions and Convergence almost everywhere

Consider a sequence of functions $\{f_k\}$ defined on a set E. If $\{f_k\}$ converges to a function f at every point on E except possibly on a set $A \subset E$ where $m(A)=0$, then $\{f_k\}$ is said to converge to f almost everywhere, abbreviated as *a.e.*, on E. The a.e. convergence is conceptually similar to the almost sure (a.s.) convergence of a random sequence.

Definition A4-19: A collection Ψ of subsets of a non-empty set Ω (which may be finite or countably infinite or uncountable) is said to be an algebra in Ω if Ψ satisfies the following properties:

(i) $\Omega \in \Psi$.

(ii) If $E \in \Psi$, then $E^c \in \Psi$ where $E^c \equiv \Omega - E$.

(iii) $E = \bigcup_{i=1}^{n} E_i \in \Psi$ where $E_i \in \Psi$ $\forall$i, then $E \in \Psi$. [This property is as closure under finite union.]

If condition (iii) is relaxed to countable union, i.e., if $E = \bigcup_{i=1}^{\infty} E_i \in \Psi$, then Ψ is called a σ-algebra in Ω.

The duple (Ω, Ψ) is called a *measurable space* where the members of Ψ are called measurable sets in Ω. When there is no confusion, we say Ω is a measurable space instead of (Ω, Ψ). ♦

Remark A4-20: The last two conditions imply that any finite (countable) intersections of events is also an event for an algebra (σ-algebra). ♦

Remark A4-21: In the terminology of probability theory, the non-empty set Ω is called the sample space which is the set of all possible outcomes (of a random experiment) or sample points, and $E \in \Psi$ is called an event. In general, the σ-algebra Ψ is called the event space which is the collection of all possible events. It should be obvious from the above three conditions that any arbitrary subset of Ω may not be qualified as an event. However, the sample space Ω (which is the sure event) and its complement in Ω, namely the empty set $\varnothing$, (which is called the impossible event) are always qualified as events. Every event space must contain these two events. Therefore, for a given sample space, the event space may not be unique. So, the smallest event space which can be obtained as the intersection of all possible event spaces is $\{\varnothing, \Omega\}$. ♦

Remark A4-22: If Ω is a finite set, then there can be only finitely many event spaces, each of which must also be a finite set. In other words, there can be only finitely many different algebras if there are only finitely many elements in Ω. The largest possible event space is the power set 2^{Ω}. However, if the cardinality of Ω is 1, i.e., if there is exactly one experimental outcome, then the only possible event space is $\{\varnothing, \Omega\}$. ♦

Remark A4-23: If Ω is an infinite set, then Ψ can be finite or infinite. This follows from the facts that the smallest Ψ is always finite and the largest Ψ is the power set 2^{Ω} which is infinite if Ω is infinite. Note that, for an infinite Ω, countable or uncountable, it is possible to construct an uncountable Ψ but there does not exist a countably infinite Ψ. ♦

Remark A4-24: It follows from De Morgan's theorem and the last two conditions of σ-algebra that any countable intersection of events is also an event, i.e., if $E = \bigcap_{i=1}^{\infty} E_i \in \Psi$ if $E_i \in \Psi$. ♦

Remark A4-25: In the context of probability theory, each event (i.e., element of the event space) is a measurable set. ♦

Definition A4-20: A nonnegative finitely additive set function μ defined on E is called finite iff μ(Ω) is finite. This implies that μ(E) is finite for every $\mathrm{E} \in \mathsf{E}$. Furthermore, μ defined on E is called σ-finite iff there exist a sequence $\{\mathrm{E_i}\}$ with $\mathrm{E_i} \in \mathsf{E}$ such that $\Omega = \bigcup_{i=1}^{\infty} \mathrm{E_i}$ and $\mu(\mathrm{E_i}) < \infty \; \forall i$. (Note that Lebesgue measure is σ-finite but not finite.). ♦