# Two-Dimensional Inverse Dynamics

**Kinematically driven**

**Must use centroidal coordinate frames !**

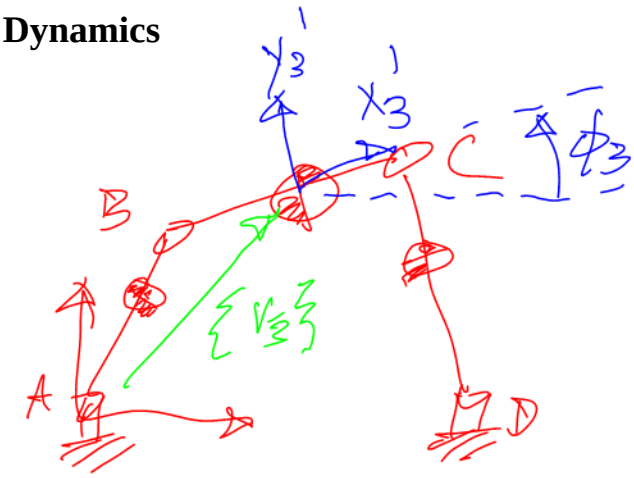
pos $\{q_i\} = \begin{Bmatrix} \{r_i\} \\ \phi_i \end{Bmatrix}$

accel $\{\ddot{q}_i\} = \begin{Bmatrix} \{\ddot{r}_i\} \\ \ddot{\phi}_i \end{Bmatrix}$

$\{Q_{on\ i}\} = \begin{Bmatrix} \{F_{on\ i}\} \\ T_{on\ i} \end{Bmatrix}$

$[M_i] = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & J_{Gi} \end{bmatrix}$

$\{Q\}_{GRAVITY} = \begin{Bmatrix} 0 \\ -m_i g \\ 0 \end{Bmatrix}$




$\{q_i\} = \begin{Bmatrix} \{r_i\} \\ \phi_i \end{Bmatrix}$

$\{Q_{on\ i}\} = \begin{Bmatrix} \{F_{on\ i}\} \\ T_{on\ i} \end{Bmatrix}$

**Single body**

$[M_i]\{\ddot{q}_i\} = \sum\left(\{Q_{on\ i}\}_{ALL}\right)$

3x3 3x1

$\{Q_{on\ i}\}_{ALL}$ $\{Q_{on\ i}\}_{APPLIED}$ $\{Q_{on\ i}\}_{CONSTRAINT}$

moment of each force $\{F_{on\ i}\}$ about the origin

**System of multiple bodies**

$$\{q\} = \begin{Bmatrix} \{q_2\} \\ \{q_3\} \\ \{q_4\} \end{Bmatrix}$$

$\{q\}$ $\{\ddot{q}\}$ $[M]$

9x1 9x1 9x9

$$\{\ddot{q}\} = \begin{Bmatrix} \{\ddot{q}_2\} \\ \{\ddot{q}_3\} \\ \{\ddot{q}_4\} \end{Bmatrix}$$

$$[M] = \begin{bmatrix} [M_2] & [0_{3x3}] & [0_{3x3}] \\ [0_{3x3}] & [M_3] & [0_{3x3}] \\ [0_{3x3}] & [0_{3x3}] & [M_4] \end{bmatrix}$$

$$\begin{bmatrix} [M_2] & [0_{3x3}] & [0_{3x3}] \\ [0_{3x3}] & [M_3] & [0_{3x3}] \\ [0_{3x3}] & [0_{3x3}] & [M_4] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}_2\} \\ \{\ddot{q}_3\} \\ \{\ddot{q}_4\} \end{Bmatrix} = \begin{Bmatrix} \sum\left(\{Q_{on\ 2}\}_{ALL}\right) \\ \sum\left(\{Q_{on\ 3}\}_{ALL}\right) \\ \sum\left(\{Q_{on\ 4}\}_{ALL}\right) \end{Bmatrix}$$

$$\{Q\}_{ALL} = \begin{Bmatrix} \sum\left(\{Q_{on\ 2}\}_{ALL}\right) \\ \sum\left(\{Q_{on\ 3}\}_{ALL}\right) \\ \sum\left(\{Q_{on\ 4}\}_{ALL}\right) \end{Bmatrix}$$

$$\boxed{[M]\{\ddot{q}\} = \{Q\}_{ALL}}$$

$$\{Q\}_{ALL} = \{Q\}_{APPLIED} + \{Q\}_{CONSTRAINT}$$

$$\{\Phi\}_{CONSTRAINT} = \begin{Bmatrix} \{\Phi\}_{KINEMATIC} \\ \{\Phi\}_{DRIVER} \end{Bmatrix}$$

$\{Q\}_{CONSTRAINT}$ $\{Q\}_{KINEMATIC}$ $\{Q\}_{DRIVER}$

$$\left.\begin{Bmatrix} \{\Phi\}_{REVA} \\ \{\Phi\}_{REVB} \\ \{\Phi\}_{REVC} \\ \{\Phi\}_{PRIS} \end{Bmatrix}\right\} \text{JOINTS}$$

$$\{\Phi\}_{DRIVER}$$

DRIVERS prescribed motion

**Virtual work**

$$[M]\{\ddot{q}\} - \{Q\}_{ALL} = \{0\}$$

$$\{\dot{q}\}^T([M]\{\ddot{q}\} - \{Q\}_{ALL}) = 0$$

$$\{Q\}_{ALL} = \{Q\}_{APPLIED} + \{Q\}_{CONSTRAINT}$$

$$\{\dot{q}\}^T\{Q\}_{CONSTRAINT} = 0 \qquad \text{for kinematic consistency} \qquad \boxed{\lfloor\Phi_q\rfloor\{\dot{q}\} = \{0\}}$$

Kinematic consistency

$$\boxed{\{\dot{q}\}^T([M]\{\ddot{q}\} - \{Q\}_{APPLIED}) = 0} \qquad \text{subject to} \qquad \lfloor\Phi_q\rfloor\{\dot{q}\} = \{0\}$$

removes $\{Q\}_{CONSTRAINT}$

**Virtual work for one revolute**

$$\{\dot{q}\}^T\{Q\}_{CONSTRAINT} =? \quad 0$$

$$\{\dot{q}_i\} = \begin{Bmatrix}\{\dot{r}_i\}\\ \dot{\phi}_i\end{Bmatrix} \qquad \{\dot{q}_j\} = \begin{Bmatrix}\{\dot{r}_j\}\\ \dot{\phi}_j\end{Bmatrix}$$

$$\{F_{on\ i}\}^P_{REV} = -\{F_{on\ j}\}^P_{REV}$$

$$\{Q_{on\ i}\}^P_{REV} = \begin{Bmatrix}\{F_{on\ i}\}^P_{REV}\\ ([B_i]\{s_i\}^{,P})^T\{F_{on\ i}\}^P_{REV}\end{Bmatrix} \qquad \{Q_{on\ j}\}^P_{REV} = \begin{Bmatrix}\{F_{on\ j}\}^P_{REV}\\ ([B_j]\{s_j\}^{,P})^T\{F_{on\ j}\}^P_{REV}\end{Bmatrix}$$

$$\{\dot{q}_i\}^T\{Q_{on\ i}\}^P_{REV} + \{\dot{q}_j\}^T\{Q_{on\ j}\}^P_{REV} =? \quad 0$$

$$\begin{Bmatrix}\{\dot{r}_i\}\\ \dot{\phi}_i\end{Bmatrix}^T\begin{Bmatrix}\{F_{on\ i}\}^P_{REV}\\ ([B_i]\{s_i\}^{,P})^T\{F_{on\ i}\}^P_{REV}\end{Bmatrix} + \begin{Bmatrix}\{\dot{r}_j\}\\ \dot{\phi}_j\end{Bmatrix}^T\begin{Bmatrix}\{F_{on\ j}\}^P_{REV}\\ ([B_j]\{s_j\}^{,P})^T\{F_{on\ j}\}^P_{REV}\end{Bmatrix} =? \quad 0$$

$$\{\dot{r}_i\}^T\{F_{on\ i}\}^P_{REV} + (\dot{\phi}_i[B_i]\{s_i\}^{,P})^T\{F_{on\ i}\}^P_{REV} + \{\dot{r}_j\}^T\{F_{on\ j}\}^P_{REV} + (\dot{\phi}_j[B_j]\{s_j\}^{,P})^T\{F_{on\ j}\}^P_{REV} =? \quad 0$$

$$\{\dot{r}_i\}^P = \{\dot{r}_i\} + \dot{\phi}_i[B_i]\{s_i\}^{,P} \qquad \{\dot{r}_j\}^P = \{\dot{r}_j\} + \dot{\phi}_j[B_j]\{s_j\}^{,P} \qquad \{\dot{r}_i\}^P = \{\dot{r}_j\}^P$$

$$(\{\dot{r}_i\}^P)^T\{F_{on\ i}\}^P_{REV} + (\{\dot{r}_j\}^P)^T\{F_{on\ j}\}^P_{REV} =? \quad 0 \qquad \textbf{OK}$$

**Lagrange multiplier theorem**

general problem $\{b\}^T\{x\}=0$ subject to $[A]\{x\}=\{0\}$

$\{b\}^T\{x\}+\{\lambda\}^T[A]\{x\}=0$ using Lagrange multipliers $\{\lambda\}$ for any arbitrary $\{x\}$

virtual work $\{\dot{q}\}^T([M]\{\ddot{q}\}-\{Q\}_{APPLIED})=0$ subject to $\lfloor\Phi_q\rfloor\{\dot{q}\}=\{0\}$

$\{x\}=\{\dot{q}\}$ $\{b\}=[M]\{\ddot{q}\}-\{Q\}_{APPLIED}$ $[A]=\lfloor\Phi_q\rfloor$

$([M]\{\ddot{q}\}-\{Q\}_{APPLIED})^T\{\dot{q}\}+\{\lambda\}^T[\Phi_q]\{\dot{q}\}=0$ for arbitrary size but kinematically consistent $\{\dot{q}\}$

$([M]\{\ddot{q}\}-\{Q\}_{APPLIED}+[\Phi_q]^T\{\lambda\})^T\{\dot{q}\}=0$

joint "forces"

$$\boxed{[M]\{\ddot{q}\}+[\Phi_q]^T\{\lambda\}=\{Q\}_{APPLIED}}$$

distribution

$[\Phi_q]^T\{\lambda\}$ each row in $\{\lambda\}$ is multiplied times corresponding column in $[\Phi_q]^T$

each row in $\{\lambda\}$ corresponds to matching row in $\lfloor\Phi_q\rfloor$ and $\{\Phi\}$

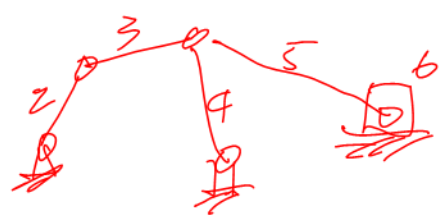
$$\boxed{[\Phi_q]^T\{\lambda\}=\{Q\}}$$

**Lagrange multipliers**

$[M]\{\ddot{q}\}+[\Phi_q]^T\{\lambda\}=\{Q\}_{APPLIED}$

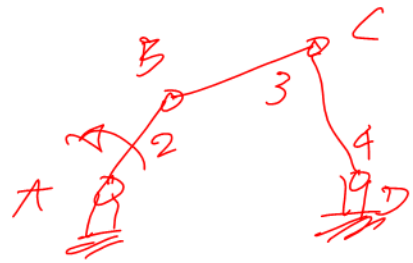
$[M]\{\ddot{q}\}=\{Q\}_{APPLIED}-[\Phi_q]^T\{\lambda\}$

$[M]\{\ddot{q}\}=\{Q\}_{ALL}=\{Q\}_{APPLIED}+\{Q\}_{CONSTRAINTS}$




$\{Q\}_{CONSTRAINTS}=-[\Phi_q]^T\{\lambda\}$

$$\{Q\}_{ALL}=\begin{Bmatrix}\sum(\{Q_{on\ 2}\}_{ALL})\\ \sum(\{Q_{on\ 3}\}_{ALL})\\ \sum(\{Q_{on\ 4}\}_{ALL})\end{Bmatrix} \qquad \{Q\}_{CONSTRAINTS}=\begin{Bmatrix}\sum(\{Q_{on\ 2}\}_{CONSTRAINTS})\\ \sum(\{Q_{on\ 3}\}_{CONSTRAINTS})\\ \sum(\{Q_{on\ 4}\}_{CONSTRAINTS})\end{Bmatrix}$$




## Equations of motion (EOM)

$$\boxed{\begin{aligned}&[M]\{\ddot{q}\}+[\Phi_q]^T\{\lambda\}=\{Q\}_{APPLIED}\\&[\Phi_q]\{\ddot{q}\}=\{\gamma\}\end{aligned}}$$

$m=1$

$nc=9$

$nq=9$

$[\Phi_q^T]$

$nc \times nq$

| $\{q\}$ | $\{\ddot{q}\}$ | $[M]$ | $\{Q\}_{APPLIED}$ | $\{\Phi\}$ | $\{\gamma\}$ | $\{\lambda\}$ | $[\Phi_q]$ |
|---|---|---|---|---|---|---|---|
| nq x 1 | nq x 1 | nq x nq | nq x 1 | nc x 1 | nc x 1 | nc x 1 | nc x nq |

nq = number of generalized coordinates
nk = number of kinematic constraints
nd = number of driver constraints
nc = total number of constraints (nc = nk + nd)

$nc=nq$

## Inverse dynamics – kinematically driven

solve kinematics $\{\ddot{q}\}=[\Phi_q]^{-1}\{\gamma\}$ $[\Phi_q]$ must have full rank nc = nq

solve for $\{\ddot{q}\}$

compute constraint forces $\{\lambda\}=\left([\Phi_q]^T\right)^{-1}\left(\{Q\}_{APPLIED}-[M]\{\ddot{q}\}\right)$

solve for $\{\lambda\}$

$$\{\Phi\}=\begin{Bmatrix}\{\Phi\}_{KINEMATIC}\\\{\Phi\}_{DRIVER}\end{Bmatrix}\qquad\{\lambda\}=\begin{Bmatrix}\{\lambda\}_{KINEMATIC}\\\{\lambda\}_{DRIVER}\end{Bmatrix}$$

for every row in $\{\Phi\}$ there is a corresponding row in $\{\lambda\}$

## Inverse dynamics – simultaneous EOM matrix

$$\boxed{[M]\{\ddot{q}\}+[\Phi_q]^T\{\lambda\}=\{Q\}_{APPLIED}}\quad\text{and}\quad\boxed{[\Phi_q]\{\ddot{q}\}=\{\gamma\}}$$

$$\begin{bmatrix}\underset{nq\times nq}{[M]} & \underset{nq\times nc}{[\Phi_q]^T}\\ \underset{nc\times nq}{[\Phi_q]} & \underset{nc\times nc}{[0]}\end{bmatrix}\begin{Bmatrix}\underset{nq\times 1}{\{\ddot{q}\}}\\ \underset{nc\times 1}{\{\lambda\}}\end{Bmatrix}=\begin{Bmatrix}\underset{nq\times 1}{\{Q\}_{APPLIED}}\\ \underset{nc\times 1}{\{\gamma\}}\end{Bmatrix}\qquad[EOM]=\underset{(nc+nq)\times(nc+nq)}{\begin{bmatrix}[M] & [\Phi_q]^T\\ [\Phi_q] & [0]\end{bmatrix}}$$

$$\begin{Bmatrix}\{\ddot{q}\}\\\{\lambda\}\end{Bmatrix}=\begin{bmatrix}[M] & [\Phi_q]^T\\ [\Phi_q] & [0]\end{bmatrix}^{-1}\begin{Bmatrix}\{Q\}_{APPLIED}\\\{\gamma\}\end{Bmatrix}$$

solve for $\{\ddot{q}\}$ and $\{\lambda\}$ simultaneously

**Statics**

$$\{\ddot{q}\} = \{0\}$$

$$[M]\{\ddot{q}\} + [\Phi_q]^T \{\lambda\} = \{Q\}_{APPLIED}$$

$$\{\lambda\} = \left([\Phi_q]^T\right)^{-1} \{Q\}_{APPLIED}$$

**Lagrange multipliers for specific constraints**

$$\{F_{on\ i}\}_{CONSTRAINT} = -[\Phi_{r\ i}]^T_{CONSTRAINT} \{\lambda\}_{CONSTRAINT}$$

$$(T_{on\ i})_{CONSTRAINT} = \left(([B_i]\{s_i\}'^P)^T [\Phi_{r\ i}]^T_{CONSTRAINT} - [\Phi_{\phi\ i}]^T_{CONSTRAINT}\right) \{\lambda\}_{CONSTRAINT}$$

$$\{F_{on\ j}\}_{CONSTRAINT} = -[\Phi_{r\ j}]^T_{CONSTRAINT} \{\lambda\}_{CONSTRAINT}$$

$$(T_{on\ j})_{CONSTRAINT} = \left(([B_j]\{s_j\}'^P)^T [\Phi_{r\ j}]^T_{CONSTRAINT} - [\Phi_{\phi\ j}]^T_{CONSTRAINT}\right) \{\lambda\}_{CONSTRAINT}$$

**Revolute**

$$\{\Phi\}_{REV} = \{r_j\}^P - \{r_i\}^P = \{0_{2x1}\}$$

**Note:** Haug uses $\{r_i\}^P - \{r_j\}^P$

$$\{F_{on\ i}\}_{REV} = \{\lambda\}_{REV} \qquad \text{at } \{r_i\}^P$$

$$(T_{on\ i})_{REV} = 0$$

$$\{F_{on\ j}\}_{REV} = -\{\lambda\}_{REV} \qquad \text{at } \{r_j\}^P$$

$$(T_{on\ j})_{REV} = 0$$

**check body i** $\quad [\Phi_{r\ i}]^P_{REV} = -[I_2] \qquad [\Phi_{\phi\ i}]^P_{REV} = -[B_i]\{s_i\}'^P$ **OK**

**check body j** $\quad [\Phi_{r\ j}]^P_{REV} = [I_2] \qquad [\Phi_{\phi\ j}]^P_{REV} = [B_j]\{s_j\}'^P$ **OK**

**Double revolute**

$$\Phi_{REV_REV} = \{d_{ij}\}^T \{d_{ij}\} - L^2 = 0 \qquad L = \text{constant length}$$

for $\{d_{ij}\} = \{r_j\}^P - \{r_i\}^P$

and $\{\dot{d}_{ij}\} = \{\dot{r}_j\}^P - \{\dot{r}_i\}^P$

and $\{\ddot{d}_{ij}\} = \{\ddot{r}_j\}^P - \{\ddot{r}_i\}^P$

and $\{\dddot{d}_{ij}\} = \{\dddot{r}_j\}^P - \{\dddot{r}_i\}^P$

$$\{F_{on\ i}\}_{REV_REV} = 2\{d_{ij}\}\lambda_{REV_REV} \qquad \text{at } \{r_i\}^P$$

$$(T_{on\ i})_{CONSTRAINT} = 0$$

$$\{F_{on\ j}\}_{REV_REV} = -2\{d_{ij}\}\lambda_{REV_REV} \qquad \text{at } \{r_j\}^P$$

$$(T_{on\ j})_{CONSTRAINT} = 0$$

**Parallel vectors (planar parallel-1)**

$\{a_i\}$ parallel to $\{a_j\}$

$$\Phi_{PARALLEL} = \{a_i\}^T [R]^T \{a_j\} = 0$$

for $\{a_i\} = \{r_i\}^Q - \{r_i\}^P$ and $\{a_j\} = \{r_j\}^Q - \{r_j\}^P$

$$\{F_{on\ i}\}_{PARALLEL} = \{0_{2x1}\}$$

$$(T_{on\ i})_{PARALLEL} = (\text{norm}\{a_i\})(\text{norm}\{a_j\})\lambda_{PARALLEL}$$

$$\{F_{on\ j}\}_{PARALLEL} = \{0_{2x1}\}$$

$$(T_{on\ j})_{PARALLEL} = -(\text{norm}\{a_i\})(\text{norm}\{a_j\})\lambda_{PARALLEL}$$

**Pin-in-slot (planar parallel-2)**

$\{a_i\}$ parallel to $\{d_{ij}\}$

$$\Phi_{PIN_SLOT} = \{a_i\}^T [R]^T \{d_{ij}\} = 0$$

for $\{d_{ij}\} = \{r_j\}^P - \{r_i\}^P$ and $\{a_i\} = \{r_i\}^Q - \{r_i\}^P$

and $\{\dot{d}_{ij}\}$ $\{\ddot{d}_{ij}\}$ $\{\dddot{d}_{ij}\}$ from above

$$\{F_{on\ i}\}_{PIN_SLOT} = [R]\{a_i\}\lambda_{PIN_SLOT} \qquad \text{at } \{r_j\}^P$$

$$(T_{on\ i})_{CONSTRAINT} = (\text{norm}\{a_i\})(\text{norm}\{d_{ij}\})\lambda_{PIN_SLOT}$$

$$\{F_{on\ j}\}_{PIN_SLOT} = -[R]\{a_i\}\lambda_{PIN_SLOT} \qquad \text{at } \{r_j\}^P$$

$$(T_{on\ j})_{PIN_SLOT} = 0$$

**<u>Relative angle driver</u>**

$$\Phi_{ANGLE} = \phi_j - \phi_i - C - f(t) = 0 \qquad C = \text{constant}$$

$$\{F_{on\ i}\}_{ANGLE} = \{0_{2x1}\}$$

$$(T_{on\ i})_{ANGLE} = \lambda_{ANGLE}$$

$$\{F_{on\ j}\}_{ANGLE} = \{0_{2x1}\}$$

$$(T_{on\ j})_{ANGLE} = -\lambda_{ANGLE}$$

**<u>Gear pair (chain/sprockets, belt/pulleys)</u>**

$$\Phi_{GEAR} = \phi_j - K\phi_i - C = 0 \qquad K = \text{constant},\ C = \text{constant}$$

external gears $K = -\rho_i / \rho_j$, internal gears $K = +\rho_i / \rho_j$

$$\{F_{on\ i}\}_{GEAR} = \{0_{2x1}\}$$

$$(T_{on\ i})_{CONSTRAINT} = K\,\lambda_{GEAR}$$

$$\{F_{on\ j}\}_{GEAR} = \{0_{2x1}\}$$

$$\left(T_{on\ j}\right)_{CONSTRAINT} = -\lambda_{GEAR}$$

**<u>Gear pair on rotating link k</u>**

$$\Phi_{GEAR_ON_K} = \left(\phi_j - \phi_k\right) - K\left(\phi_i - \phi_k\right) - C = 0 \qquad K = \text{constant},\ C = \text{constant} \quad \text{from above}$$

$$\left\{F_{on\ i}\right\}_{GEAR} = \left\{0_{2x1}\right\}$$

$$\left(T_{on\ i}\right)_{CONSTRAINT} = K\ \lambda_{GEAR}$$

$$\left\{F_{on\ j}\right\}_{GEAR} = \left\{0_{2x1}\right\}$$

$$\left(T_{on\ j}\right)_{CONSTRAINT} = -\lambda_{GEAR}$$

$$\left\{F_{on\ k}\right\}_{GEAR} = \left\{0_{2x1}\right\}$$

$$\left(T_{on\ k}\right)_{CONSTRAINT} = \left(1 - K\right)\lambda_{GEAR}$$