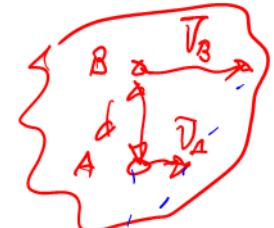


## Two-Dimensional Experimental Kinematics

Digitize locations of landmarks  $\{r_i\}^{Pk}$  on body  $i$  for points  $k=1$  to  $n$  at given time  $t$

All points must be attached to body  $i$



Use landmark weighting factor  $f^{Pk} = 1$  if point  $k$  is available at time  $t$ . Use  $f^{Pk} = 0$  if point  $k$  not available at given time  $t$ .

$$\omega \approx \frac{v_B - v_A}{d}$$

Determine  $\{\dot{r}_i\}^{Pk}$ ,  $\{\ddot{r}_i\}^{Pk}$ ,  $\{\dddot{r}_i\}^{Pk}$ ,  $\{\ddot{\ddot{r}}_i\}^{Pk}$  at given time  $t$ .

### Mean values

$$\{r_i\}^{\text{mean}} = \left( \sum_{k=1}^n f^{Pk} \{r_i\}^{Pk} \right) / \sum_{k=1}^n f^{Pk}$$

*centroid*

$$\{\dot{r}_i\}^{\text{mean}} = \left( \sum_{k=1}^n f^{Pk} \{\dot{r}_i\}^{Pk} \right) / \sum_{k=1}^n f^{Pk}$$

*mean vel*

$$\{\ddot{r}_i\}^{\text{mean}} = \left( \sum_{k=1}^n f^{Pk} \{\ddot{r}_i\}^{Pk} \right) / \sum_{k=1}^n f^{Pk}$$

*mean acc*

$$\{\ddot{\ddot{r}}_i\}^{\text{mean}} = \left( \sum_{k=1}^n f^{Pk} \{\ddot{\ddot{r}}_i\}^{Pk} \right) / \sum_{k=1}^n f^{Pk}$$

*mean jerk*

$$\{\ddot{\ddot{\ddot{r}}}_i\}^{\text{mean}} = \left( \sum_{k=1}^n f^{Pk} \{\ddot{\ddot{\ddot{r}}}_i\}^{Pk} \right) / \sum_{k=1}^n f^{Pk}$$

*mean snap*

$$S = \sum_{k=1}^n \left( f^{Pk} (\{r_i\}^{Pk} - \{r_i\}^{\text{mean}})^T (\{r_i\}^{Pk} - \{r_i\}^{\text{mean}}) \right)$$

### Velocity

$$\omega_i = \left( \sum_{k=1}^n f^{Pk} (\{\dot{r}_i\}^{Pk})^T [R] (\{r_i\}^{Pk} - \{r_i\}^{\text{mean}}) \right) / S \quad \text{for} \quad [R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$[\tilde{\omega}_i] = \omega_i [R]$$

Point ICR is the instantaneous center of rotation for body  $i$  with respect to the ground. Note that the ICR is not attached to the body. Point  $P$  on the body coincident with ICR has zero velocity.

$$\{\dot{r}_i\}^P = [\tilde{\omega}_i] (\{r_i\}^P - \{r_i\}^{\text{ICR}}) \quad \text{for any point } P \text{ attached to body } i$$

$$\{r_i\}^{\text{ICR}} = \{r_i\}^{\text{mean}} - [\tilde{\omega}_i]^{-1} \{\dot{r}_i\}^{\text{mean}} \quad \text{for} \quad \{\dot{r}_i\}^{P \text{ at ICR}} = 0$$

## Acceleration

$$\dot{\omega}_i = \left( \sum_{k=1}^n f^{Pk} \left( \{ \ddot{r}_i \}^{Pk} \right)^T [R] \left( \{ r_i \}^{Pk} - \{ r_i \}^{\text{mean}} \right) \right) / S$$



$$[\tilde{\omega}_i] = \dot{\omega}_i [R] \quad [\beta] = [\tilde{\omega}_i] + [\tilde{\omega}_i] [\tilde{\omega}_i] = \begin{bmatrix} -\omega_i^2 & -\dot{\omega}_i \\ \dot{\omega}_i & -\omega_i^2 \end{bmatrix}$$

Point IAP is the instantaneous acceleration pole for body i. Note that the IAP is not attached to the body. Point P on the body coincident with IAP has zero acceleration.

$$\{ \ddot{r}_i \}^P = [\beta] (\{ r_i \}^P - \{ r_i \}^{\text{IAP}}) \quad \text{for any point } P \text{ attached to body } i$$

$$\{ r_i \}^{\text{IAP}} = \{ r_i \}^{\text{mean}} - [\beta]^{-1} \{ \ddot{r}_i \}^{\text{mean}} \quad \text{for} \quad \{ \ddot{r}_i \}^P \text{ at IAP} = 0 \quad \text{OLD}$$

## Jerk

$$\ddot{\omega}_i = \omega_i^3 + \left( \sum_{k=1}^n f^{Pk} \left( \{ \dddot{r}_i \}^{Pk} \right)^T [R] \left( \{ r_i \}^{Pk} - \{ r_i \}^{\text{mean}} \right) \right) / S$$

$$[\tilde{\omega}_i] = \ddot{\omega}_i [R] \quad [\eta] = [\tilde{\omega}_i] + 3[\tilde{\omega}_i] \tilde{\omega}_i + [\tilde{\omega}_i] [\tilde{\omega}_i] [\tilde{\omega}_i] = \begin{bmatrix} -3\omega_i \dot{\omega}_i & \omega_i^3 - \ddot{\omega}_i \\ \ddot{\omega}_i - \omega_i^3 & -3\omega_i \dot{\omega}_i \end{bmatrix}$$

Point IJP is the instantaneous jerk pole for the body. Note that the IJP is not attached to the body. Point P on the body coincident with IJP has zero jerk.

$$\{ \dddot{r}_i \}^P = [\eta] (\{ r_i \}^P - \{ r_i \}^{\text{IJP}}) \quad \text{for any point } P \text{ attached to the body}$$

$$\{ r_i \}^{\text{IJP}} = \{ r_i \}^{\text{mean}} - [\eta]^{-1} \{ \ddot{r}_i \}^{\text{mean}} \quad \text{for} \quad \{ \ddot{r}_i \}^P \text{ at IJP} = 0$$

## Snap

$$\ddot{\omega}_i = 6\omega_i^2 \dot{\omega}_i + \left( \sum_{k=1}^n f^{Pk} \left( \{ \ddot{r}_i \}^{Pk} \right)^T [R] \left( \{ r_i \}^{Pk} - \{ r_i \}^{\text{mean}} \right) \right) / S$$

$$[\tilde{\omega}_i] = \ddot{\omega}_i [R] \quad [\sigma] = [\tilde{\omega}_i] + 6[\tilde{\omega}_i] [\tilde{\omega}_i] [\tilde{\omega}_i] + 4[\tilde{\omega}_i] [\tilde{\omega}_i] + 3[\tilde{\omega}_i] [\tilde{\omega}_i] [\tilde{\omega}_i]$$

$$[\sigma] = \begin{bmatrix} -4\omega_i \ddot{\omega}_i - 3\dot{\omega}_i^2 + \omega_i^4 & \omega_i^2 \dot{\omega}_i - \ddot{\omega}_i \\ \ddot{\omega}_i - \omega_i^2 \dot{\omega}_i & -4\omega_i \ddot{\omega}_i - 3\dot{\omega}_i^2 + \omega_i^4 \end{bmatrix}$$

Point ISP is the instantaneous snap pole for the body. Note that the ISP is not attached to the body. Point P on the body coincident with ISP has zero snap.

$$\{\ddot{r}\}^P = [\sigma] \{r_i\}^P - \{r\}^{ISP} \quad \text{for any point P attached to the body}$$

$$\{r\}^{ISP} = \{r_i\}^{\text{mean}} - [\sigma]^{-1} \{\ddot{r}_i\}^{\text{mean}} \quad \text{for} \quad \{\ddot{r}\}^{P \text{ at } ISP} = 0$$

## Centrode

Location of the ICR measured relative to ground changes as body i moves and sweeps a locus called the fixed centrode for body i. The time derivative of the locus describes how the ICR moves. Tracking the location of the ICR relative to a coordinate frame fixed to body i provides a locus called the moving centrode. Motion of body i may be characterized as pure rolling of the moving centrode on the fixed centrode because body i instantaneously has zero velocity at each location of the ICR.

$$\{\dot{r}\}^{ICR} = (\tilde{\omega} \{\ddot{r}_i\}^{\text{mean}} - [\beta] \{\dot{r}_i\}^{\text{mean}}) / \omega_i^2$$

The second time derivative of the location of the ICR also changes as body i moves. First and second time derivatives of position along a locus may be combined to determine curvature  $\kappa$  of the fixed centrode. If body i is part of a mechanism with mobility of one, curvature of the centrode at each location will be invariant to speed of the mechanism.

$$\{\ddot{r}\}^{ICR} = \left( \begin{bmatrix} 0 & \omega_i \ddot{\omega}_i - 2\dot{\omega}_i^2 \\ 2\dot{\omega}_i^2 - \omega_i \ddot{\omega}_i & 0 \end{bmatrix} \{\dot{r}_i\}^{\text{mean}} + \begin{bmatrix} \omega_i^3 & 2\omega_i \dot{\omega}_i \\ -2\omega_i \dot{\omega}_i & \omega_i^3 \end{bmatrix} \{\ddot{r}_i\}^{\text{mean}} + \begin{bmatrix} 0 & -\omega_i^2 \\ \omega_i^2 & 0 \end{bmatrix} \{\dot{r}_i\}^{\text{mean}} \right) / \omega_i^3$$

$$\kappa^{ICR} = \left( (\{\dot{r}\}^{ICR})^T [R] \{\dot{r}\}^{ICR} \right) / \left( (\{\dot{r}\}^{ICR})^T \{\dot{r}\}^{ICR} \right)^{3/2}$$

## Relative velocity

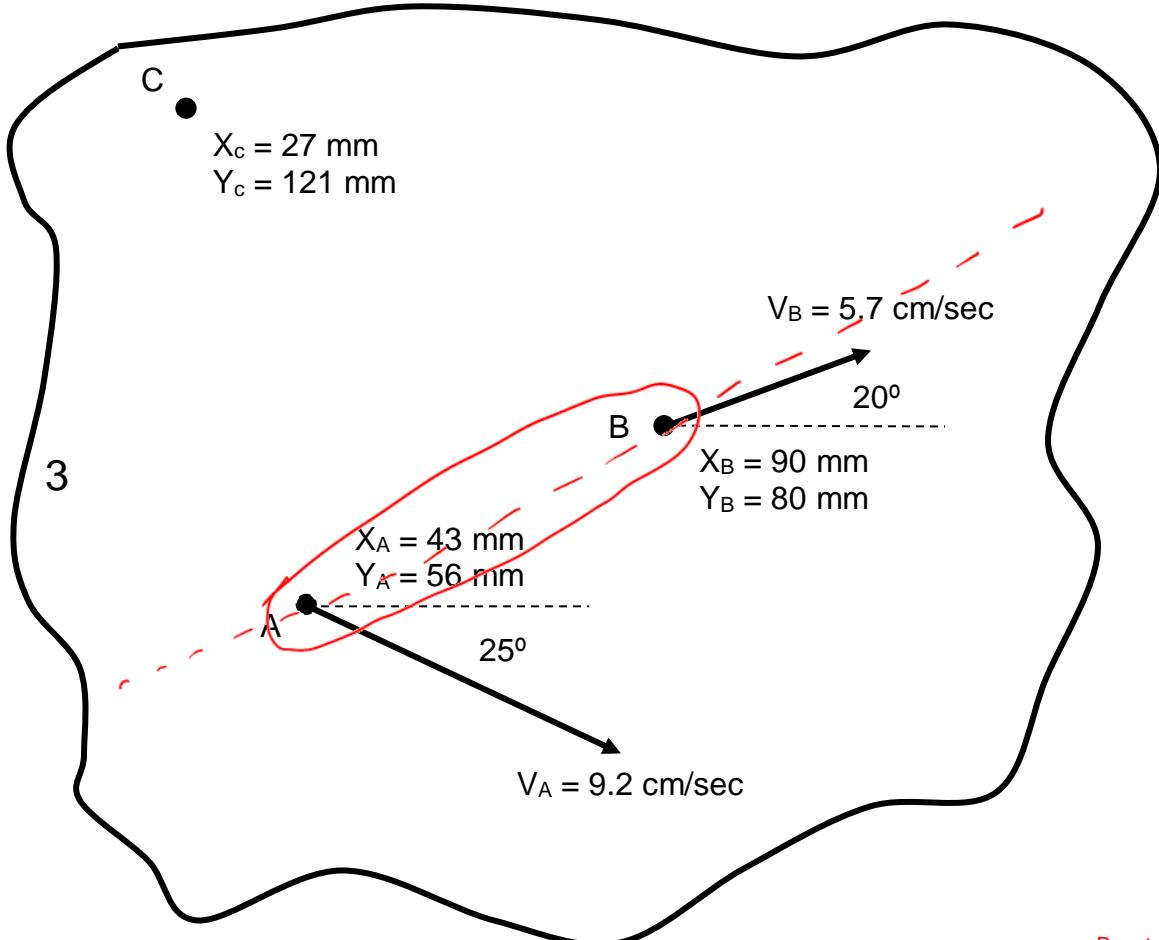
For planar motion, the relative angular velocity of body j with respect to body i is the difference between the two angular velocities. The relative instantaneous center of rotation (RICR) for body j about body i describes a unique point that has zero relative velocity between the two bodies. Note that location of the RICR is measured with respect to the ground.

$$\omega_{j\_wrt\_i} = \omega_j - \omega_i$$

$$\{\dot{r}_{j\_wrt\_i}\}^{ICR} = (\tilde{\omega}_j \{\dot{r}_j\}^{ICR} - \tilde{\omega}_i \{\dot{r}_i\}^{ICR}) / \omega_{j\_wrt\_i}$$

## Rigid Body

Determine the velocity of point C on rigid body link 3. The rigid body and the velocity vectors are drawn to scale. Link 3 is NOT pinned to the ground. Show your work.



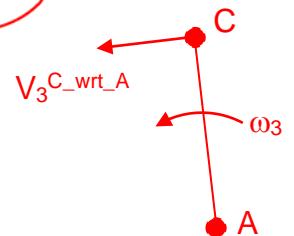
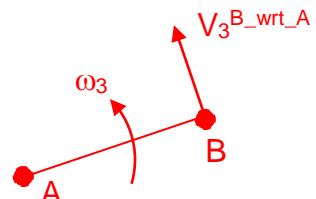
$$\{\dot{r}_3\}^B = \begin{Bmatrix} 53.56 \text{ mmmps} \\ 19.50 \text{ mmmps} \end{Bmatrix} \quad \{\dot{r}_3\}^A = \begin{Bmatrix} 83.38 \text{ mmmps} \\ -38.88 \text{ mmmps} \end{Bmatrix}$$

$$\{\dot{r}_3\}^{B\_wrt\_A} = \{\dot{r}_3\}^B - \{\dot{r}_3\}^A = \begin{Bmatrix} -29.82 \text{ mmmps} \\ 58.38 \text{ mmmps} \end{Bmatrix} = 65.55 \text{ mmmps} \angle 117.1^\circ$$

$$\text{norm}\{\dot{r}_3\}^{B\_wrt\_A} = \omega_3(AB) \quad AB = 52.77 \text{ mm} \quad \omega_3 = 1.242 \text{ rad/sec CCW}$$

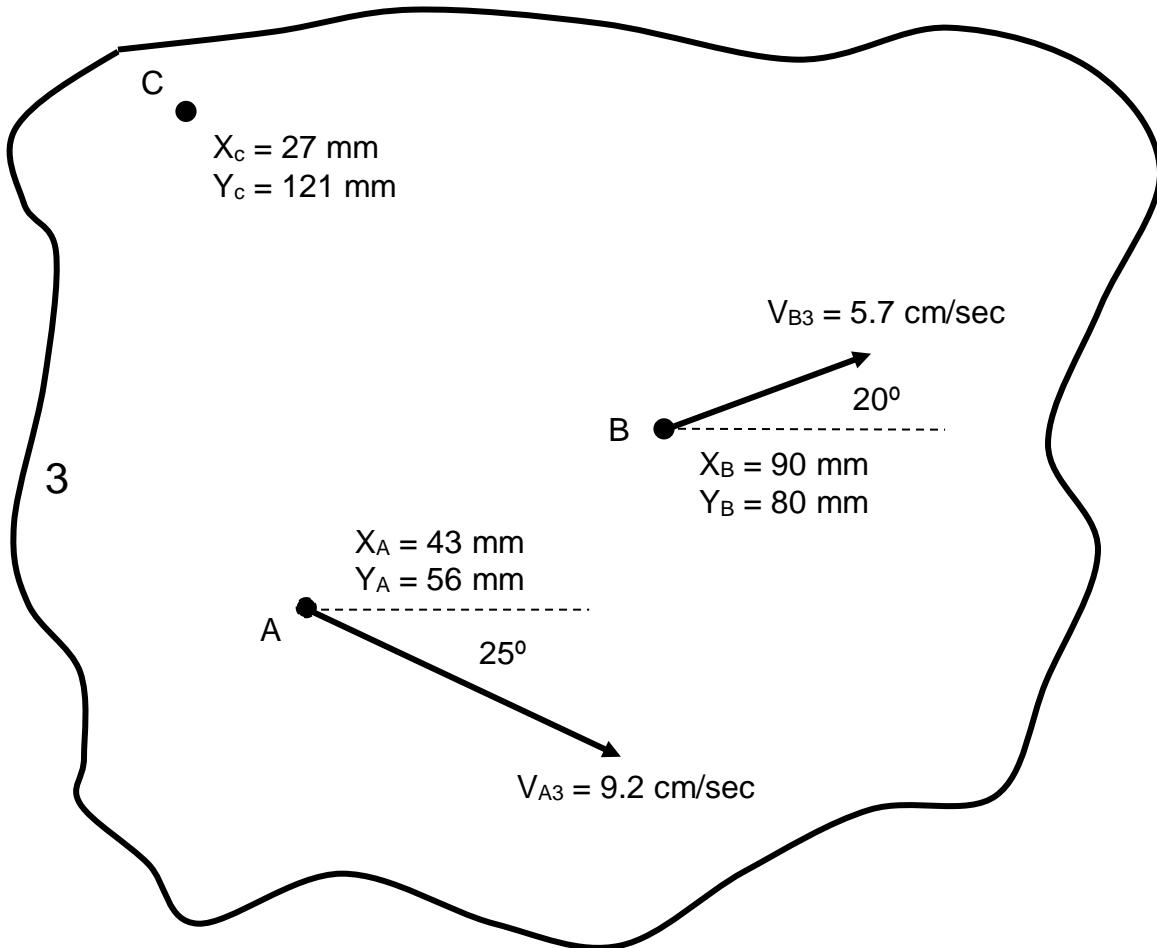
$$\{\dot{r}_3\}^{C\_wrt\_A} = \begin{Bmatrix} -16 \text{ mm} \\ 65 \text{ mm} \end{Bmatrix} \quad \{\dot{r}_3\}^{C\_wrt\_A} = \omega_3[R]\{\dot{r}_3\}^{B\_wrt\_A} = \begin{Bmatrix} -80.73 \text{ mmmps} \\ -19.87 \text{ mmmps} \end{Bmatrix}$$

$$\boxed{\{\dot{r}_3\}^C = \{\dot{r}_3\}^A + \{\dot{r}_3\}^{C\_wrt\_A} = \begin{Bmatrix} 2.65 \text{ mmmps} \\ -58.75 \text{ mmmps} \end{Bmatrix} = 58.81 \text{ mmmps} \angle -87.4^\circ}$$



## Rigid Body

Determine the velocity of point C on rigid body link 3. The rigid body and the velocity vectors are drawn to scale. Link 3 is NOT pinned to the ground. Show your work.



$$\text{point A} = P1 \quad f^{P1} = 1 \quad \{r_3\}^{P1} = \begin{Bmatrix} 43 \text{ mm} \\ 56 \text{ mm} \end{Bmatrix} \quad \{\dot{r}_3\}^{P1} = \begin{Bmatrix} 83.38 \text{ mmmps} \\ -38.88 \text{ mmmps} \end{Bmatrix} \text{ from above}$$

$$\text{point B} = P2 \quad f^{P2} = 1 \quad \{r_3\}^{P2} = \begin{Bmatrix} 90 \text{ mm} \\ 80 \text{ mm} \end{Bmatrix} \quad \{\dot{r}_3\}^{P2} = \begin{Bmatrix} 53.56 \text{ mmmps} \\ 19.50 \text{ mmmps} \end{Bmatrix}$$

use lm2kin2d per attached code

$$\omega_3 = +1.2422 \text{ rad/sec} \quad \{r\}^{\text{ICR}} = \begin{Bmatrix} 74.3006 \\ 123.1197 \end{Bmatrix} \text{ mm}$$

$$\{r_3\}^C = \begin{Bmatrix} 27 \text{ mm} \\ 121 \text{ mm} \end{Bmatrix} \quad \{\dot{r}_3\}^C = \begin{bmatrix} 0 & -\omega_3 \\ \omega_3 & 0 \end{bmatrix} (\{r_3\}^C - \{r\}^{\text{ICR}}) = \begin{Bmatrix} 2.6332 \text{ mmmps} \\ -58.7571 \text{ mmmps} \end{Bmatrix} = 58.82 \text{ mmmps} \angle -87.43^\circ$$

```
% rbk.m - rigid body kinematics
% HJSIII, 14.01.13

clear

% constants
Rmat = [ 0 -1 ; 1 0 ];

% inputs
f = [ 1 1 ];

r = [ 43 90 ;
      56 80 ];

rd = [ 83.38 53.56 ;
       -38.88 19.5 ];

rdd = zeros(2,2);

rddd = zeros(2,2);

% call function
[ w, rICR, wd, rIAP, wdd, rIJP, rdICR, kappa ] = lm2kin2d( f, r, rd, rdd, rddd );
w
rICR

% find velocity of C
r3C = [ 27 121 ]';
r3Cd = w * Rmat * ( r3C - rICR )

% bottom of rbk
```

```
% t_lm2kin2d.m - test 2D kinematics from landmark motion
% HJSIII, 14.01.13

clear

%%%%%%%%%%%%%
% example inputs - web cutter four bar
% Haug page 197 - not ME 581 web cutter
%          B3      C3
f = [    1       1     ];
r = [ 3.7588   3.9407 ;
      1.3681  29.3675 ];
rd = [ -5.4874  22.5296 ;
      15.0764 14.8943 ];
rdd = [ -60.4716 -42.8075 ;
      -22.0098 -50.1604 ];
rddd = [  88.2815 -673.2083 ;
      -242.5514 -291.1768 ];

% expected outputs
w_test =      -1.0006;
rICR_test = [ 18.8257 ; 6.8520 ];

wd_test =      -0.6374;
rIAP_test = [ -49.1784 ; 13.0846 ];

wdd_test =      26.1823;
rIJP_test = [ 12.8647 ; 3.9747 ];

rdICR_test = [ -37.0807 ; 72.0169 ];
kappa_test =      -0.0151 ;
%%%%%%%%%%%%%

% test function
[ w, rICR, wd, rIAP, wdd, rIJP, rdICR, kappa ] = lm2kin2d( f, r, rd, rdd, rddd );
w
rICR
wd
rIAP
wdd
rIJP
rdICR
kappa

% bottom of t_lm2kin2d
```

```

function [ w, rICR, wd, rIAP, wdd, rIJP, rdICR, kappa ] = lm2kin2d( f, r, rd, rdd, rddd )
% 2D instantaneous kinematics of a rigid body from landmark motion
% HJSIII, 14.01.13
%
% USAGE
% function [ w, rICR, wd, rIAP, wdd, rIJP, rdICR, kappa ] = lm2kin2d( f, r, rd, rdd, rddd )

% INPUTS
% f - 1xn vector of weights - f(j)=1 means data valid, f(j)=0 means data not available
% r - 2xn matrix of x,y landmark location
% rd - 2xn matrix of x,y landmark velocity
% rdd - 2xn matrix of x,y landmark acceleration
% rddd - 2xn matrix of x,y landmark jerk
%
% OUTPUTS
% rICR - 2x1 location of instantaneous center of rotation
% w - angular velocity
% rIAP - 2x1 location of instantaneous acceleration pole
% wd - angular acceleration
% rIJP - 2x1 location of instantaneous jerk pole
% wdd - angular jerk
% rdICR - 2x1 time derivative of location of instantaneous center
% kappa - curvature of centrode

% constants
Rmat = [ 0 -1 ; 1 0 ];

% mean values
[ nr, n ] = size( r );
fmat = diag( f );
sf = sum( f' );
rm = sum( fmat*r' )' /sf;
rdm = sum( fmat*rd' )' /sf;
rddm = sum( fmat*rdd' )' /sf;
rddd = sum( fmat*rddd' )' /sf;

% centered location
rc = r - rm*ones(1,n);
S = trace( rc * fmat * rc' );

% velocity
vmat = rd * fmat * rc';
w = ( vmat(2,1) - vmat(1,2) ) /S;
wsk = w * Rmat;
rICR = rm - inv(wsk) * rdm;

% acceleration
amat = rdd * fmat * rc';
wd = ( amat(2,1) - amat(1,2) ) /S;
wdsk = wd * Rmat;
beta = wdsk + wsk*wsk;
rIAP = rm - inv(beta) * rddm;

% jerk
jmat = rddd * fmat * rc';
wdd = w*w*w + ( jmat(2,1) - jmat(1,2) ) /S;
wddsk = wdd * Rmat;
eta = wddsk + 3*wsk*wdsk + wsk*wsk*wsk;
rIJP = rm - inv(eta) * rddd;

% snap
%rddd = sum( fmat*rddd' )' /sf;
%smat = rddd * fmat * rc';
%wddd = 6*w*w*wd + ( smat(2,1) - smat(1,2) ) /S;
%wddsk = wdd * Rmat;
%sigma = wddsk + 6*wsk*wsk*wdsk + 4*wsk*wddsk + 3*wdsk*wdsk + wsk*wsk*wsk*wsk;
%rlSP = rm - inv(sigma) * rddd;

% centrode
rdICR = ( wsk*rddm - beta*rdm ) /w/w;
nrdICR = norm( rdICR );

sk1 = [ 0 (w*wdd-2*wd*wd) ; -(w*wdd-2*wd*wd) 0 ];

```

```
sk2 = [ w*w*w  2*w*wd ; -2*w*wd  w*w*w ];
sk3 = [ 0  -w*w ; w*w  0 ];
rddICR = ( sk1*rdm + sk2*rddm + sk3*rddd ) /w/w/w;

kappa = ( rdICR(1)*rddICR(2) - rdICR(2)*rddICR(1) ) /nrdICR/nrdICR/nrdICR;

% bottom of lm2kin2d
```