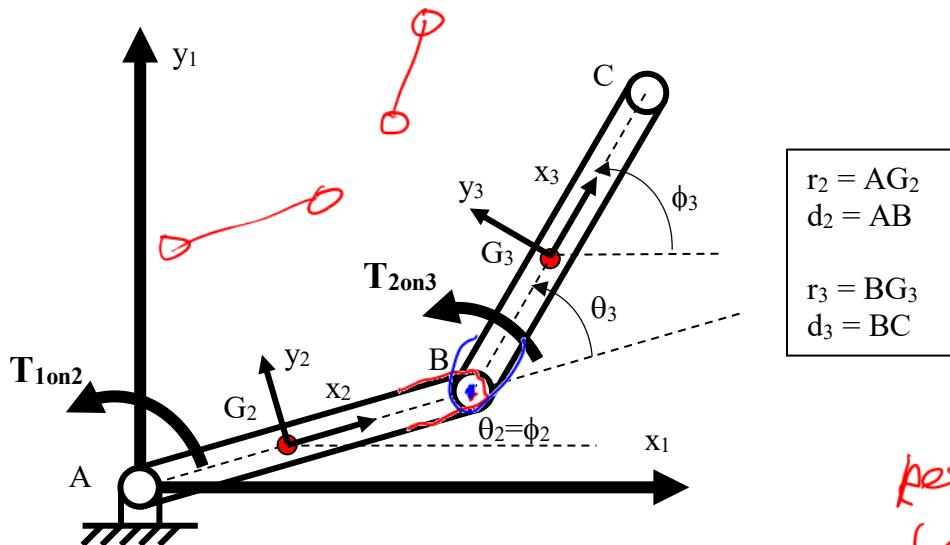


Differential-Algebraic Equations (DAE) for Anthropomorphic Manipulator



Two solid rigid bars with revolute joints A and B

Tool center point (TCP) at C (endpoint)

Centroids G_2 and G_3

Masses m_2 and m_3

Centroidal mass moments of inertia J_{G_2} and J_{G_3}

T_{1on2} is torque of ground on bar 2 about revolute A measured CCW positive

T_{2on3} is torque of bar 2 on bar 3 about revolute B measured CCW positive

Gravity g acts along negative y axis

$$nL = 3 \quad nJ1=2 \quad m = 3 \quad (nL-1) - 2 \quad nJ1 = 2$$

*keep track
of global locations
and orientations
of local frames
of attached links*

Lagrangian method (from Notes_09_02)

$$\{q\} = \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} \quad \dot{\{q\}} = \begin{Bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \quad \{Q\} = \begin{Bmatrix} T_{1 \text{ on } 2} \\ T_{2 \text{ on } 3} \end{Bmatrix} \quad \text{Note: } \theta_3 \text{ measured relative to } \theta_2$$

$$J_B = m_3 a_3^2 + J_3$$

$$J_A = J_B + m_2 a_2^2 + m_3 d_2^2 + J_2 + 2m_3 d_2 a_3 \cos \theta_3$$

$$C = J_B + m_3 d_2 a_3 \cos \theta_3$$

$$D = m_3 d_2 a_3 \sin \theta_3$$

$$G_2 = (m_2 a_2 + m_3 d_2) g \cos \theta_2$$

$$G_3 = m_3 a_3 g \cos(\theta_2 + \theta_3)$$

inverse dynamics

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} + \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix}$$

know driver motion at any time t - find $\{q\}$ $\{\dot{q}\}$ $\{\ddot{q}\}$

compute driver torques $T_{1 \text{ on } 2}$ $T_{2 \text{ on } 3}$ from Lagrangian equations

compute joint forces $F_{1 \text{ on } 2}^x$ $F_{1 \text{ on } 2}^y$ $F_{2 \text{ on } 3}^x$ $F_{2 \text{ on } 3}^y$ from Newtonian equations

may arbitrarily choose any other time t

forward dynamics

$$\begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix}^{-1} \left(\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} - \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} - \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix} \right)$$

know current state $\{q\}$ and $\{\dot{q}\}$ at current t

compute $\{\dot{q}\}$ from Lagrangian equations

compute $F_{1 \text{ on } 2}^x$ $F_{1 \text{ on } 2}^y$ $F_{2 \text{ on } 3}^x$ $F_{2 \text{ on } 3}^y$ from Newtonian equations

must integrate $\{\ddot{q}\}$ to get new $\{q\}$ and $\{\dot{q}\}$ at the next time step

Newtonian method (from free body diagrams)

$$\begin{aligned}
 F_{1\text{on}2}^x - F_{2\text{on}3}^x &= m_2 \ddot{x}_2 \\
 F_{1\text{on}2}^y - F_{2\text{on}3}^y - m_2 g &= m_2 \ddot{y}_2 \\
 \{s_2\}^A \times \{F_{1\text{on}2}\} - \{s_2\}^B \times \{F_{2\text{on}3}\} + T_{1\text{on}2} - T_{2\text{on}3} &= J_{G2} \ddot{\phi}_2 \\
 F_{2\text{on}3}^x &= m_3 \ddot{x}_3 \\
 F_{2\text{on}3}^y - m_3 g &= m_3 \ddot{y}_3 \\
 \{s_3\}^B \times \{F_{2\text{on}3}\} + T_{2\text{on}3} J_{G3} \ddot{\phi}_3 &
 \end{aligned}$$

inverse dynamics

distribution matrix

$$\left[\begin{array}{ccccc|c}
 +1 & 0 & 0 & -1 & 0 & 0 \\
 0 & +1 & 0 & 0 & -1 & 0 \\
 -(\{s_2\}^A)^Y & +(\{s_2\}^A)^X & +1 & (\{s_2\}^B)^Y & -(\{s_2\}^B)^X & -1 \\
 0 & 0 & 0 & +1 & 0 & 0 \\
 0 & 0 & 0 & 0 & +1 & 0 \\
 0 & 0 & 0 & -(\{s_3\}^B)^Y & +(\{s_3\}^B)^X & +1
 \end{array} \right] \left[\begin{array}{c}
 F_{1\text{on}2}^x \\
 F_{1\text{on}2}^y \\
 T_{1\text{on}2} \\
 F_{2\text{on}3}^x \\
 F_{2\text{on}3}^y \\
 T_{2\text{on}3}
 \end{array} \right]^T = \left[\begin{array}{c}
 m_2 \ddot{x}_2 \\
 m_2 \ddot{y}_2 + m_2 g \\
 J_{G2} \ddot{\phi}_2 \\
 m_3 \ddot{x}_3 \\
 m_3 \ddot{y}_3 + m_3 g \\
 J_{G3} \ddot{\phi}_3
 \end{array} \right]$$

know driver motion at any time t - find positions, velocities and accelerations

compute joint forces $\{F_{1\text{on}2}^x \quad F_{1\text{on}2}^y \quad F_{2\text{on}3}^x \quad F_{2\text{on}3}^y \quad T_{1\text{on}2} \quad T_{2\text{on}3}\}^T$

may arbitrarily choose any other time t

forward dynamics

know current positions and velocities at current t

very cumbersome to compute joint forces and accelerations simultaneously

must integrate accelerations to get new positions and velocities at the next time step

*pivot forces
and driver
"forces"*

*mass x accel
ext forces*

DAE dynamics

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$$\{q\} = \begin{bmatrix} x_2 \\ y_2 \\ \phi_2 \\ x_3 \\ y_3 \\ \phi_3 \end{bmatrix} \quad \{s_2\}^A = \begin{cases} -AG_2 \\ 0 \end{cases} \quad \{s_2\}^B = \begin{cases} BG_2 \\ 0 \end{cases}$$

*local locations
of joints*

$$\{\Phi\}_{\text{KINEMATIC}} = \begin{cases} \{r_2\}^A - \{r_1\}^A \\ \{r_3\}^B - \{r_2\}^B \end{cases} \quad \{\Phi\}_{\text{DRIVERS}} = \begin{cases} f_1(\{q\}, t) \\ f_2(\{q\}, t) \end{cases}$$

*Er₃^B - Er₂^B = 0
inverse dynamics
driver constraints
= mobility*

*{s₃} constant
blueprint*

*Er₂^A + global
location of A on 2*

*Sources
all to zero*

by hand

$$\left[\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{12 \times 12} \right] \left\{ \begin{array}{l} \{\ddot{q}\} \\ \{\lambda\} \end{array} \right\}_{12 \times 1} = \left\{ \begin{array}{l} \{Q\}_{\text{EXT}} \\ \{\gamma\} \end{array} \right\}_{12 \times 1}$$

$$\{\Phi\} = \left\{ \begin{array}{l} \{r_2\}^A - \{r_1\}^A \\ \{r_3\}^B - \{r_2\}^B \end{array} \right\}_{\text{DRIVERS}} \quad \{\lambda\} = \begin{cases} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \\ T_{1 \text{ on } 2} \\ T_{2 \text{ on } 3} \end{cases}$$

{F₃} internal joint forces

know driver motion at any time t, find $\{q\} \quad \{\dot{q}\} \quad \{\gamma\} \quad \{Q\}_{\text{EXT}} \quad [\Phi_q]$

compute $\{\ddot{q}\}$ and $\{\lambda\}$ simultaneously

may arbitrarily choose any other time t

local RHS *Jacobian*

*[M]
diagonal
of masses*

forward dynamics

$$\left[\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{10 \times 10} \right] \left\{ \begin{array}{l} \{\ddot{q}\} \\ \{\lambda\} \end{array} \right\}_{10 \times 1} = \left\{ \begin{array}{l} \{Q\}_{\text{EXT}} \\ \{\gamma\} \end{array} \right\}_{10 \times 1} \quad \{\Phi\} = \begin{cases} \{r_2\}^A - \{r_1\}^A \\ \{r_3\}^B - \{r_2\}^B \end{cases} \quad \{\lambda\} = \begin{cases} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \end{cases}$$

know current state $\{q\}$ and $\{\dot{q}\}$ at current t, find $\{\gamma\} \quad \{Q\}_{\text{EXT}} \quad [\Phi_q]$

compute $\{\ddot{q}\}$ and $\{\lambda\}$ simultaneously

must integrate $\{\ddot{q}\}$ to get new $\{q\}$ and $\{\dot{q}\}$ at the next time step

$\dot{\phi}_2 = \omega_2 t$

$\dot{\phi}_3 = \omega_3 t$

$\dot{\phi}_4 = \omega_4 t$

$\{q\} = \begin{Bmatrix} x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$

$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \\ \{\Phi\}_{REV_C} \\ \{\Phi\}_{REV_D} \end{Bmatrix}$

Notes_08_13

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Inverse Dynamics – joint interpolated motion (independent position controllers on each joint)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \\ \{\Phi\}_{REV_C} \\ \{\Phi\}_{REV_D} \end{Bmatrix}$$

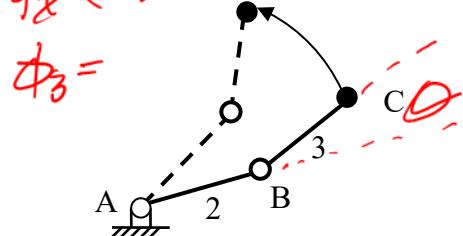
$\boxed{\Phi_2 - \Phi_{2_START} - \omega_2 t}$

$\boxed{\theta - \theta_{START} - \omega_0 t}$

$$\omega_2 = (\phi_{2_END} - \phi_{2_START}) / \Delta t$$

$$\theta = \phi_3 - \phi_2$$

$$\omega_0 = (\theta_{END} - \theta_{START}) / \Delta t$$



$$\{\gamma\}^A - \{\gamma\}^B = \{d\}$$

$$[\Phi_q]_{6x6} \quad \{v\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

rows correspond to $\{\Phi\}$
columns correspond to $\{\ddot{q}\}$

$$\{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV_A} \\ \{\gamma\}_{REV_B} \\ 0 \\ 0 \end{Bmatrix}$$

rows correspond to $\{v\}$
columns correspond to $\{\gamma\}$

$$\{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

external forces

$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}$$

$$\{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \\ T_{1 \text{ on } 2} \\ T_{2 \text{ on } 3} \end{Bmatrix}$$

$$\dot{\phi}_2 - \dot{\phi}_{2_START} - \omega_2 t - \frac{1}{2} \alpha_2 t^2$$

Inverse Dynamics – straight-line TCP interpolated motion (interpolated position controllers on each joint)

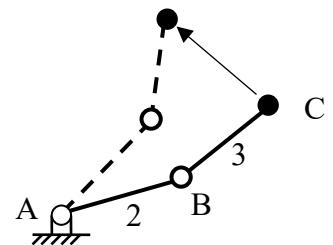
$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \\ \{\Phi\}_{REV_C} \\ \{\Phi\}_{REV_D} \end{Bmatrix}$$

$\boxed{x_{C3} - x_{C3_START} - v_{C3_x} t}$

$\boxed{y_{C3} - y_{C3_START} - v_{C3_y} t}$

$$v_{C3_x} = (x_{C3_END} - x_{C3_START}) / \Delta t$$

$$v_{C3_y} = (y_{C3_END} - y_{C3_START}) / \Delta t$$



$$[\Phi_q]_{6x6} \quad \{v\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \\ v_{C3_x} \\ v_{C3_y} \end{Bmatrix}$$

$$\{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV_A} \\ \{\gamma\}_{REV_B} \\ 0 \\ 0 \end{Bmatrix}$$

$$\{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}$$

$$\{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \\ F_{EFF_CX} \\ F_{EFF_CY} \end{Bmatrix}$$

$$\theta = \phi_3 - \phi_2 \quad \{r_3\}^C = \begin{cases} r_2 \cos \phi_2 + r_3 \cos(\phi_2 + \theta) \\ r_2 \sin \phi_2 + r_3 \sin(\phi_2 + \theta) \end{cases}$$

$$\{\dot{r}_3\}^C = \begin{bmatrix} -r_2 \sin \phi_2 - r_3 \sin(\phi_2 + \theta) & -r_3 \sin(\phi_2 + \theta) \\ r_2 \cos \phi_2 r_3 \cos(\phi_2 + \theta) & r_3 \cos(\phi_2 + \theta) \end{bmatrix} \begin{bmatrix} \dot{\phi}_2 \\ \dot{\theta} \end{bmatrix}$$

$$\begin{Bmatrix} F_{EFF_CX} \\ F_{EFF_CY} \end{Bmatrix}^T \{\dot{r}_3\}^C + \begin{Bmatrix} T_{1on2} \\ T_{2on3} \end{Bmatrix}^T \begin{Bmatrix} \dot{\phi}_2 \\ \dot{\theta} \end{Bmatrix} = 0$$

$$\begin{Bmatrix} T_{1on2} \\ T_{2on3} \end{Bmatrix} = - \begin{bmatrix} -r_2 \sin \phi_2 - r_3 \sin(\phi_2 + \theta) & -r_3 \sin(\phi_2 + \theta) \\ r_2 \cos \phi_2 r_3 \cos(\phi_2 + \theta) & r_3 \cos(\phi_2 + \theta) \end{bmatrix}^T \begin{Bmatrix} F_{EFF_CX} \\ F_{EFF_CY} \end{Bmatrix}$$

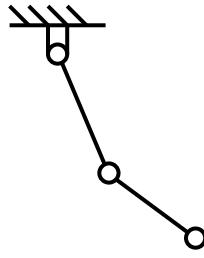
Forward Dynamics – double pendulum

(no actuators)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \end{Bmatrix}$$

Ax1

ND
drivers



$$\boxed{\begin{bmatrix} \Phi_q \\ 4 \times 6 \end{bmatrix}}$$

$$\{v\} = \begin{Bmatrix} \{0_{2 \times 1}\} \\ \{0_{2 \times 1}\} \end{Bmatrix}$$

Ax1

$$\{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV_A} \\ \{\gamma\}_{REV_B} \end{Bmatrix}$$

Ax1

$$\{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

$$\{f_B\} = \begin{Bmatrix} x_2 \\ y_2 \\ \dot{\phi}_2 \\ x_3 \\ y_3 \\ \dot{\phi}_3 \end{Bmatrix}$$

$$\boxed{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{10 \times 10} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}_{10 \times 1} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}_{10 \times 1}} \quad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1onA2} \\ \{F\}_{B2onB3} \end{Bmatrix}$$

$$\begin{bmatrix} \boxed{[M]} & \boxed{[\Phi_q]^T} \\ \boxed{[\Phi_q]} & \boxed{[0]} \end{bmatrix}_{10 \times 10} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}_{10 \times 1} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}_{10 \times 1}$$

10x10

10x1

10x1

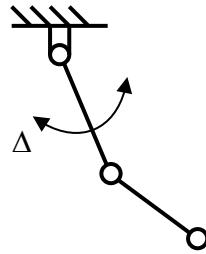
$$\begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}_{10 \times 1} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}_{10 \times 1}$$

10x1

Forward Dynamics – proximal link kinematically driven, distal link pendulum
 (position controller only on proximal joint)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \\ \phi_2 - \phi_{2_CENTER} - \Delta \sin(2\pi f t) \end{Bmatrix}$$

5x1



$$\begin{bmatrix} \Phi_q \\ 5x6 \end{bmatrix} \quad \{v\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \\ 2\pi f \Delta \cos(2\pi f t) \end{Bmatrix}$$

5x6

5x1

$$\{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV_A} \\ \{\gamma\}_{REV_B} \\ -4\pi^2 f^2 \Delta \sin(2\pi f t) \end{Bmatrix} \quad \{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

5x1

$$\boxed{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{11x11} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix}_{11x1} = \begin{bmatrix} Q_{EXT} \\ \gamma \end{bmatrix}_{11x1}}$$

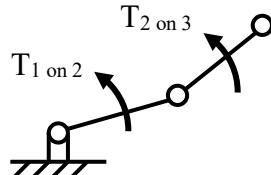
$$\{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \\ T_{1 \text{ on } 2} \end{Bmatrix}$$

6x1

Forward Dynamics – computed torque control
 (torque controllers on each joint)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \end{Bmatrix}$$

no drivers



$$\begin{bmatrix} \Phi_q \\ 4x6 \end{bmatrix} \quad \{v\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \end{Bmatrix}$$

$$\{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV_A} \\ \{\gamma\}_{REV_B} \end{Bmatrix}$$

$$\{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ T_{1 \text{ on } 2} - T_{2 \text{ on } 3} \\ 0 \\ -m_3 g \\ T_{2 \text{ on } 3} \end{Bmatrix}$$

$$\boxed{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{10x10} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix}_{10x1} = \begin{bmatrix} Q_{EXT} \\ \gamma \end{bmatrix}_{10x1}}$$

$$\{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \end{Bmatrix}$$