

$$\{r_3\}^c = \{r_3\} + [A_3] \{s_3\}^x$$

## Two-Dimensional Constraints

### General

*position*  
axi  
 $\{\Phi\} = \{0\}$

$$[\text{JAC}] \{ \text{gen vel} \} = \{ \text{rd RHS} \}$$

*vel*  
 $\frac{d}{dt}$   
 $\dot{\{\Phi\}} = \{0\}$

$$[\dot{\Phi}_q] \{ \ddot{q} \} = - \{ \dot{\Phi}_t \}$$

*acc*  
 $[\Phi_q] \{ \ddot{q} \} = \{ v \}$

$$\{ v \} = - \{ \dot{\Phi}_t \}$$

$$\{ \ddot{q} \} = [\dot{\Phi}_q] \{ \ddot{v} \}$$

$$[\dot{\Phi}_t]_{\text{KIN}} = \{ 0 \}$$

*JOINTS*

*acc*  
 $\ddot{\{\Phi\}} = \{0\}$

*JAC*    *gen accel*    *RHS*  
 $[\Phi_q] \{ \ddot{q} \} = \{ \gamma \}$

$$\{ \gamma \} = - ([\Phi_q] \{ \dot{q} \})_q \{ \dot{q} \} - 2[\Phi_{qt}] \{ \dot{q} \} - \{ \Phi_{tt} \}$$

$\{ \ddot{\Phi} \} = \{ 0 \}$

$$\{ \ddot{\Phi} \} = [\dot{\Phi}_q] \{ \ddot{q} \}$$

$[\Phi_q] \{ \ddot{q} \} = \{ \eta \}$

$\{ \eta \} = -3([\Phi_q] \{ \dot{q} \})_q \{ \dot{q} \} - ([\Phi_q] \{ \dot{q} \})_q \{ \dot{q} \} - 3[\Phi_{qt}] \{ \dot{q} \} - 3([\Phi_{qt}] \{ \dot{q} \})_q \{ \dot{q} \} - 3[\Phi_{ttt}] \{ \dot{q} \} - \{ \Phi_{ttt} \}$

$\{ \ddot{\Phi} \} = \{ 0 \}$

$[\Phi_q] \{ \ddot{q} \} = \{ \sigma \}$

$\{ \sigma \} = -4([\Phi_q] \{ \ddot{q} \})_q \{ \dot{q} \} - 3([\Phi_q] \{ \ddot{q} \})_q \{ \dot{q} \} - 6([\Phi_q] \{ \ddot{q} \})_q \{ \dot{q} \} - \left( ([\Phi_q] \{ \ddot{q} \})_q \{ \dot{q} \} \right)_q \{ \dot{q} \}$

$$- 4[\Phi_{qt}] \{ \ddot{q} \} - 12([\Phi_{qt}] \{ \ddot{q} \})_q \{ \dot{q} \} - 4([\Phi_{qt}] \{ \ddot{q} \})_q \{ \dot{q} \} - 6[\Phi_{ttt}] \{ \ddot{q} \} - 6([\Phi_{ttt}] \{ \ddot{q} \})_q \{ \dot{q} \}$$

$$- 4[\Phi_{ttt}] \{ \dot{q} \} - \{ \Phi_{tttt} \}$$

$\{ \ddot{\Phi} \}_{\text{KIN}} = - ([\dot{\Phi}_q] \{ \ddot{q} \}) \{ \ddot{q} \}$

*look up below*

### Scleronomic constraints

independent of time such as mechanical joints

$$\{\gamma\} \equiv -(\Phi_q \dot{q})_q \dot{q}$$

$$\{\eta\} \equiv \{\ddot{\gamma}\} - (\Phi_q \ddot{q})_q \dot{q}$$

$$\{\sigma\} \equiv \{\dot{\eta}\} - (\Phi_q \ddot{q})_q \dot{q}$$

### Revolute

$$\{\Phi\}_{REV} = \{r_j\}^P - \{r_i\}^P = \{0_{2x1}\}$$

$$\begin{aligned} \text{2x3} \quad [\Phi_{qi}]_{REV} &= -[ [I_2] \quad [B_i] \{s_i\}^P ] & \text{2x3} \\ \text{2x3} \quad [\Phi_{qj}]_{REV} &= [ [I_2] \quad [B_j] \{s_j\}^P ] & \text{2x3} \end{aligned}$$

$$\text{2x1} \quad \{v\}_{REV} = \{0_{2x1}\}$$

$$\text{2x1} \quad \{\gamma\}_{REV} = \dot{\phi}_j^2 [A_j] \{s_j\}^P - \dot{\phi}_i^2 [A_i] \{s_i\}^P$$

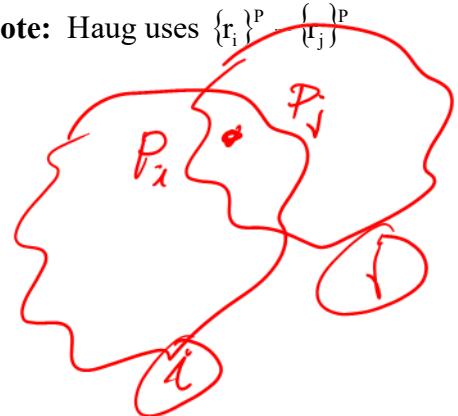
$$\{\eta\}_{REV} = \dot{\phi}_j^3 [B_j] \{s_j\}^P + 3 \dot{\phi}_j \ddot{\phi}_j [A_j] \{s_j\}^P - \dot{\phi}_i^3 [B_i] \{s_i\}^P - 3 \dot{\phi}_i \ddot{\phi}_i [A_i] \{s_i\}^P$$

$$\begin{aligned} \{\sigma\}_{REV} &= 6 \dot{\phi}_j^2 \ddot{\phi}_j [B_j] \{s_j\}^P + (4 \dot{\phi}_j \ddot{\phi}_j + 3 \ddot{\phi}_j^2 - \dot{\phi}_j^4) [A_j] \{s_j\}^P \\ &\quad - 6 \dot{\phi}_i^2 \ddot{\phi}_i [B_i] \{s_i\}^P - (4 \dot{\phi}_i \ddot{\phi}_i + 3 \ddot{\phi}_i^2 - \dot{\phi}_i^4) [A_i] \{s_i\}^P \end{aligned}$$

$$\{\kappa_i\}_{REV} = \left\{ \begin{array}{c} [0_{2x1}] \\ \{s_i\}^{PT} [A_i]^T \{\lambda\}_{REV} \dot{\phi}_i \end{array} \right\}$$

$$\{\kappa_j\}_{REV} = \left\{ \begin{array}{c} [0_{2x1}] \\ -\{s_j\}^{PT} [A_j]^T \{\lambda\}_{REV} \dot{\phi}_j \end{array} \right\}$$

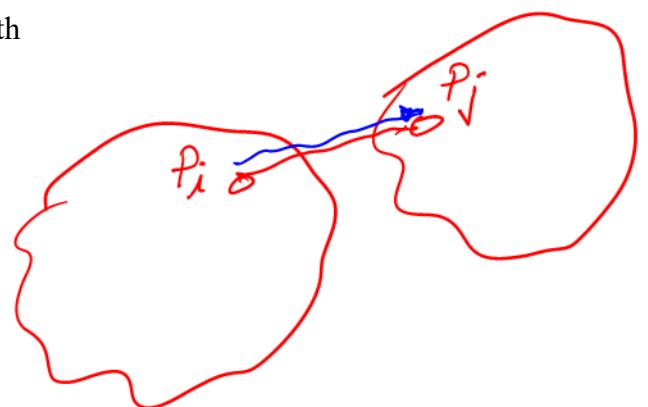
**Note:** Haug uses  $\{r_i\}^P - \{r_j\}^P$



### Double revolute

N)  $\Phi_{REV\_REV} = \{d_{ij}\}^T \{d_{ij}\} - L^2 = 0$        $L = \text{constant length}$

for  $\{d_{ij}\} = \{r_j\}^p - \{r_i\}^p$   
and  $\{\dot{d}_{ij}\} = \{\dot{r}_j\}^p - \{\dot{r}_i\}^p$   
and  $\{\ddot{d}_{ij}\} = \{\ddot{r}_j\}^p - \{\ddot{r}_i\}^p$   
and  $\{\dddot{d}_{ij}\} = \{\ddot{r}_j\}^p - \{\ddot{r}_i\}^p$



1x3  $[\Phi_{qi}]_{REV\_REV} = 2\{d_{ij}\}^T [\Phi_{qi}]_{REV}$       1x7 2x3

1x3  $[\Phi_{qi}]_{REV\_REV} = 2\{d_{ij}\}^T [\Phi_{qi}]_{REV}$

1x1  $v_{REV\_REV} = 0$

1x1  $\gamma_{REV\_REV} = 2\{d_{ij}\}^T \{\gamma\}_{REV} - 2\{\dot{d}_{ij}\}^T \{\dot{d}_{ij}\}$

$\eta_{REV\_REV} = 2\{d_{ij}\}^T \{\eta\}_{REV} - 6\{\dot{d}_{ij}\}^T \{\ddot{d}_{ij}\}$

$\sigma_{REV\_REV} = 2\{d_{ij}\}^T \{\sigma\}_{REV} - 8\{\dot{d}_{ij}\}^T \{\ddot{d}_{ij}\} - 6\{\ddot{d}_{ij}\}^T \{\ddot{d}_{ij}\}$

$$\{\kappa_i\}_{REV\_REV} = -2\lambda_{REV\_REV} \left\{ \begin{array}{l} \{\dot{d}_{ij}\} \\ \{s_i\}^{IP^T} \left( [B_i]^T \{\dot{d}_{ij}\} - [A_i]^T \{d_{ij}\} \right) \dot{\phi}_i \end{array} \right\}$$

$$\{\kappa_j\}_{REV\_REV} = 2\lambda_{REV\_REV} \left\{ \begin{array}{l} \{\dot{d}_{ij}\} \\ \{s_j\}^{IP^T} \left( [B_j]^T \{\dot{d}_{ij}\} - [A_j]^T \{d_{ij}\} \right) \dot{\phi}_j \end{array} \right\}$$

### Parallel vectors (planar parallel-1)

$\{a_i\}$  parallel to  $\{a_j\}$

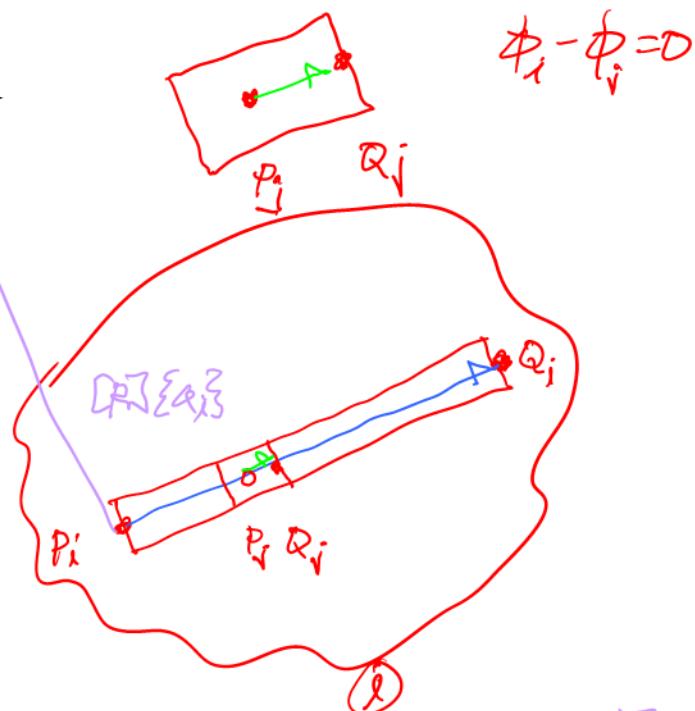
$$\boxed{R} \boxed{G} \boxed{B} \perp \boxed{a}$$

1x1  $\Phi_{PARALLEL} = \{a_i\}^T [R]^T \{a_j\} = 0$

for  $\{a_i\} = \{r_i\}^q - \{r_i\}^p$  and  $\{a_j\} = \{r_j\}^q - \{r_j\}^p$

1x3  $[\Phi_{qi}]_{PARALLEL} = \left[ \{0_{1x2}\} \quad -\{a_i\}^T \{a_j\} \right] \quad \{a_i\}^T \{a_j\} = \pm \text{norm}\{a_i\} \times \text{norm}\{a_j\} = \text{constant}$

$$\boxed{R} = \boxed{0-1}$$



$$181 \quad [\Phi_{qj}]_{\text{PARALLEL}} = [0_{1 \times 2}] + \{a_i\}^T \{a_j\}$$

$$171 \quad v_{\text{PARALLEL}} = 0$$

$$181 \quad \gamma_{\text{PARALLEL}} = 0$$

$$\eta_{\text{PARALLEL}} = 0$$

$$\sigma_{\text{PARALLEL}} = 0$$

$$\{\kappa_i\}_{\text{PARALLEL}} = \{0_{3 \times 1}\}$$

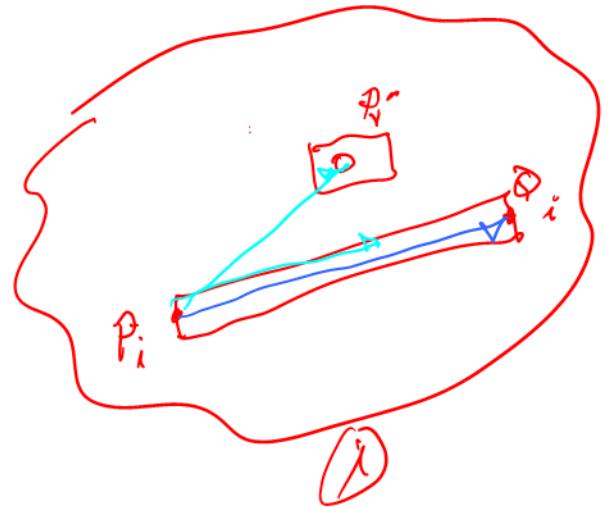
$$\{\kappa_j\}_{\text{PARALLEL}} = \{0_{3 \times 1}\}$$

### Pin-in-slot (planar parallel-2)

$\{a_i\}$  parallel to  $\{d_{ij}\}$

$$181 \quad \Phi_{\text{PIN\_SLOT}} = \{a_i\}^T [R]^T \{d_{ij}\} = 0$$

for  $\{d_{ij}\} = \{r_j\}^P - \{r_i\}^P$  and  $\{a_i\} = \{r_i\}^Q - \{r_i\}^P$   
and  $\{\dot{d}_{ij}\}, \{\ddot{d}_{ij}\}, \{\ddot{\ddot{d}}_{ij}\}$  from above



$$183 \quad [\Phi_{qi}]_{\text{PIN\_SLOT}} = \{a_i\}^T [R]^T [\Phi_{qi}]_{\text{REV}} - [0_{1 \times 2}] - \{a_i\}^T \{d_{ij}\}$$

$$183 \quad [\Phi_{qj}]_{\text{PIN\_SLOT}} = \{a_i\}^T [R]^T [\Phi_{qj}]_{\text{REV}}$$

$$181 \quad v_{\text{PIN\_SLOT}} = 0$$

$$181 \quad \gamma_{\text{PIN\_SLOT}} = \{a_i\}^T (2 \dot{\phi}_i \{\dot{d}_{ij}\} + [R]^T (\dot{\phi}_i^2 \{d_{ij}\} + \{\gamma\}_{\text{REV}}))$$

$$\eta_{\text{PIN\_SLOT}} = \{a_i\}^T (3 \dot{\phi}_i \{\ddot{d}_{ij}\} + 3 \ddot{\phi}_i \{\dot{d}_{ij}\} - \dot{\phi}_i^3 \{d_{ij}\} + [R]^T (3 \dot{\phi}_i^2 \{\dot{d}_{ij}\} + 3 \dot{\phi}_i \ddot{\phi}_i \{d_{ij}\} + \{\eta\}_{\text{REV}}))$$

$$\sigma_{\text{PIN\_SLOT}} = \{a_i\}^T \left( \begin{array}{l} 4 \dot{\phi}_i \{\ddot{d}_{ij}\} + 6 \ddot{\phi}_i \{\dot{d}_{ij}\} + 4 (\ddot{\phi}_i - \dot{\phi}_i^3) \{\dot{d}_{ij}\} - 6 \dot{\phi}_i^2 \ddot{\phi}_i \{d_{ij}\} \\ + [R]^T (6 \dot{\phi}_i^2 \{\dot{d}_{ij}\} + 12 \dot{\phi}_i \ddot{\phi}_i \{\dot{d}_{ij}\} + (4 \dot{\phi}_i \ddot{\phi}_i + 3 \dot{\phi}_i^2 - \dot{\phi}_i^4) \{d_{ij}\} + \{\sigma\}_{\text{REV}}) \end{array} \right)$$

$$\{\kappa_i\}_{\text{PIN\_SLOT}} = \lambda_{\text{PIN\_SLOT}} \left\{ \begin{array}{l} \dot{\phi}_i \{a_i\} \\ - \{\dot{d}_{ij}\}^T \{a_i\} \end{array} \right\}$$

$$\{\kappa_j\}_{PIN} = \lambda_{PIN} \left\{ -\{a_i\} \dot{\phi}_j \right. \\ \left. \left\{ s_j \right\}^T [B_j]^T \{a_i\} (\dot{\phi}_j - \dot{\phi}_i) \right\}$$

### Relative angle driver

$$\Phi_{ANGLE} = \phi_j - \phi_i - C - f(t) = 0 \quad C = \text{cons tan t}$$

$$\begin{bmatrix} \Phi_{qi} \end{bmatrix}_{ANGLE} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{ji} \end{bmatrix}_{ANGLE} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

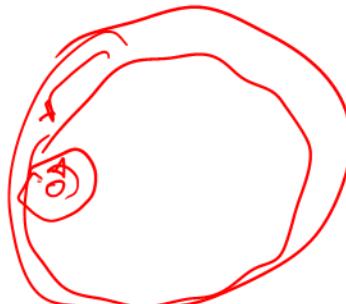
$$v_{ANGLE} = f_t$$

$$\gamma_{ANGLE} = f_{tt}$$

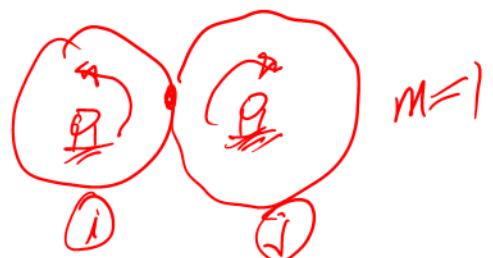
$$\eta_{ANGLE} = f_{ttt}$$

$$\{\kappa_i\}_{ANGLE} = \{0_{3x1}\}$$

$$\{\kappa_j\}_{ANGLE} = \{0_{3x1}\}$$



not  
kinematic  
constraint



### Gear pair driver (chain/sprockets, belt/pulleys)

$$\Phi_{GEAR} = \phi_j - K\phi_i - C = 0 \quad K = \text{cons tan t}, C = \text{cons tan t}$$

external gears  $K = -\rho_i / \rho_j$ , internal gears  $K = +\rho_i / \rho_j$

$$\begin{bmatrix} \Phi_{qi} \end{bmatrix}_{GEAR} = \begin{bmatrix} 0 & 0 & -K \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{qj} \end{bmatrix}_{GEAR} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$v_{GEAR} = 0$$

$$\gamma_{GEAR} = 0$$

$$\sigma_{GEAR} = 0$$

$$nL = 3$$

$$nZ_1 = 2$$

$$mZ_2 = 1$$

$$\dot{\phi}_i = K\dot{\phi}_j + C$$

### Gear pair on rotating link k

$$\Phi_{\text{GEAR\_ON\_K}} = (\phi_j - \phi_k) - K(\phi_i - \phi_k) - C = 0 \quad K = \text{const} \tan t, C = \text{const} \tan t \quad \text{from above}$$

$$[\Phi_{qi}]_{\text{GEAR\_ON\_K}} = [0 \ 0 \ -K]$$

$$[\Phi_{qj}]_{\text{GEAR\_ON\_K}} = [0 \ 0 \ 1]$$

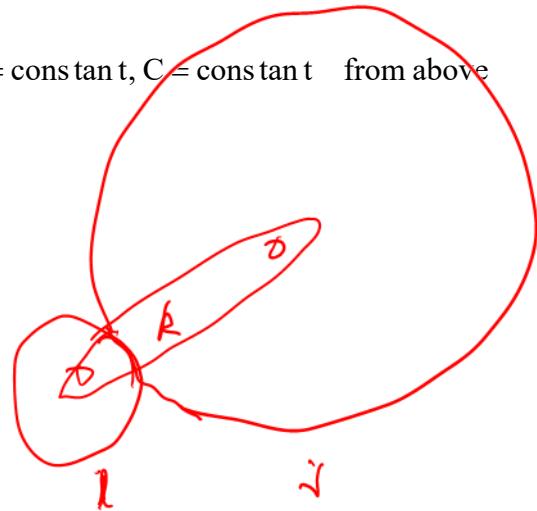
$$[\Phi_{qk}]_{\text{GEAR\_ON\_K}} = [0 \ 0 \ (K-1)]$$

$$v_{\text{GEAR\_ON\_K}} = 0$$

$$\gamma_{\text{GEAR\_ON\_K}} = 0$$

$$\eta_{\text{GEAR\_ON\_K}} = 0$$

$$\sigma_{\text{GEAR\_ON\_K}} = 0$$



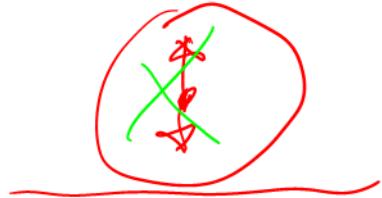
### Relative coordinate driver (translation, rotation, gears, pure rolling)

$$\Phi_{\text{RCD}} = q_j - Kq_i - C - f(t) = 0 \quad K = \text{const} \tan t, C = \text{const} \tan t$$

$$[\Phi_{qi}]_{\text{RCD}} = [-K \ 0 \ 0] \quad q_i = x_i$$

$$[\Phi_{qi}]_{\text{RCD}} = [0 \ -K \ 0] \quad q_i = y_i$$

$$[\Phi_{qi}]_{\text{RCD}} = [0 \ 0 \ -K] \quad q_i = \phi_i$$



$$[\Phi_{qj}]_{\text{RCD}} = [1 \ 0 \ 0] \quad q_j = x_j$$

$$[\Phi_{qj}]_{\text{RCD}} = [0 \ 1 \ 0] \quad q_j = y_j$$

$$[\Phi_{qj}]_{\text{RCD}} = [0 \ 0 \ 1] \quad q_j = \phi_j$$

$$v_{\text{RCD}} = f_t$$

$$\gamma_{\text{RCD}} = f_{tt}$$

$$\eta_{\text{RCD}} = f_{ttt}$$

$$\sigma_{\text{RCD}} = f_{tttt}$$

### Planar parallel-2 distance driver (see pin-in-slot)

$$\Phi_{PP2DD} = \{a_i\}^T \{d_{ij}\} / L - f(t) = 0 \quad L = |\{a_i\}| = \text{cons tan t length}$$

$$\begin{aligned} [\Phi_{qi}]_{PP2DD} &= (\{a_i\}^T [\Phi_{qi}]_{REV} + [\{0_{1x2}\} \quad \{a_i\}^T [R]^T \{d_{ij}\}]) / L \\ [\Phi_{qj}]_{PP2DD} &= \{a_i\}^T [\Phi_{qj}]_{REV} / L \end{aligned}$$

$$v_{PP2DD} = f_t$$

$$\gamma_{PP2DD} = \{a_i\}^T (-2 \dot{\phi}_i [R]^T \{d_{ij}\} + \dot{\phi}_i^2 \{d_{ij}\} + \{\gamma\}_{REV}) / L + f_{tt}$$

$$\eta_{PP2DD} = \{a_i\}^T (-[R]^T (3 \dot{\phi}_i \{d_{ij}\} + 3 \ddot{\phi}_i \{d_{ij}\} - \dot{\phi}_i^3 \{d_{ij}\}) + 3 \dot{\phi}_i^2 \{d_{ij}\} + 3 \dot{\phi}_i \ddot{\phi}_i \{d_{ij}\} + \{\eta\}_{REV}) / L + f_{ttt}$$

$$\sigma_{PP2DD} = \{a_i\}^T \left( \begin{array}{l} -[R]^T (4 \dot{\phi}_i \{d_{ij}\} + 6 \ddot{\phi}_i \{d_{ij}\} + 4 (\ddot{\phi}_i - \dot{\phi}_i^3) \{d_{ij}\} - 6 \dot{\phi}_i^2 \ddot{\phi}_i \{d_{ij}\}) \\ + 6 \dot{\phi}_i^2 \{d_{ij}\} + 12 \dot{\phi}_i \ddot{\phi}_i \{d_{ij}\} + (4\dot{\phi}_i \ddot{\phi}_i + 3\ddot{\phi}_i^2 - \dot{\phi}_i^4) \{d_{ij}\} + \{\sigma\}_{REV} \end{array} \right) / L + f_{tttt}$$

### Pure rolling along planar parallel-2 distance

$$\Phi_{ROLL} = \{a_i\}^T \{d_{ij}\} / L - \rho(\phi_j - \phi_i) - C = 0$$

$L = |\{a_i\}| = \text{cons tan t length}$ ,  $\rho = \text{rolling radius}$ ,  $C = \text{cons tan t}$

$$\begin{aligned} [\Phi_{qi}]_{ROLL} &= (\{a_i\}^T [\Phi_{qi}]_{REV} + [\{0_{1x2}\} \quad \{a_i\}^T [R]^T \{d_{ij}\} + \rho L]) / L \\ [\Phi_{qj}]_{ROLL} &= (\{a_i\}^T [\Phi_{qj}]_{REV} - [\{0_{1x2}\} \quad \rho L]) / L \end{aligned}$$

$$v_{ROLL} = 0$$

$$\gamma_{ROLL} = \gamma_{PP2DD} \quad \text{for } f_{tt} = 0$$

$$\eta_{ROLL} = \eta_{PP2DD} \quad \text{for } f_{ttt} = 0$$

$$\sigma_{ROLL} = \sigma_{PP2DD} \quad \text{for } f_{tttt} = 0$$

### Planar relative distance driver (see double revolute)

$$\Phi_{PRDD} = \{d_{ij}\}^T \{d_{ij}\} - (f(t))^2 = 0 \quad f(t) > 0$$

$$\left[ \Phi_{qi} \right]_{PRDD} = \left[ \Phi_{qi} \right]_{REV\_REV}$$

$$v_{PRDD} = 2 f f_t$$

$$\gamma_{PRDD} = \gamma_{REV\_REV} + 2 f_t^2 + 2 f f_{tt}$$

$$\eta_{PRDD} = \eta_{REV\_REV} + 6 f_t f_{tt} + 2 f f_{ttt}$$

$$\sigma_{PRDD} = \sigma_{REV\_REV} + 6 f_{tt}^2 + 8 f_t f_{ttt} + 2 f f_{ttt}$$

## Acceleration Right-hand Side for Revolute

$$\{\gamma\} \equiv -(\Phi_q \dot{q})_q \dot{q} - 2[\Phi_q]_t \dot{q} - \{\Phi_{tt}\}$$

$$\{\Phi\}_{REV} = \{r_j\}^p - \{r_i\}^p = \{0_{2x1}\}$$

$$\{q_i\} = \begin{Bmatrix} \{r_i\} \\ \phi_i \end{Bmatrix} \quad \dot{q}_i = \begin{Bmatrix} \{r_i\} \\ \dot{\phi}_i \end{Bmatrix}$$

$$[\Phi_{qi}]_{REV} = -[ [I_2] \quad [B_i] \{s_i\}^p ]$$

$$[\Phi_{qi}] \dot{q}_i = -[ [I_2] \quad [B_i] \{s_i\}^p ] \begin{Bmatrix} \{r_i\} \\ \dot{\phi}_i \end{Bmatrix} = -\{r_i\} - \dot{\phi}_i [B_i] \{s_i\}^p$$

$$([\Phi_{qi}] \dot{q}_i)_q = [ [0_{2x2}] \quad \dot{\phi}_i [A_i] \{s_i\}^p ] \quad [B_i]_{\phi i} = -[A_i]$$

$$([\Phi_{qi}] \dot{q}_i)_{qi} \dot{q}_i = [ [0_{2x2}] \quad \dot{\phi}_i [A_i] \{s_i\}^p ] \begin{Bmatrix} \{r_i\} \\ \dot{\phi}_i \end{Bmatrix} = \dot{\phi}_i^2 [A_i] \{s_i\}^p$$

$$[\Phi_{qi}]_{REV} = [ [I_2] \quad [B_i] \{s_i\}^p ]$$

$$[\Phi_{qi}]_t = [0_{2x3}] \quad [\Phi_{qi}]_t \dot{q}_i = \{0_{2x1}\}$$

$$\{\Phi_t\} = \{0_{2x1}\} \quad \{\Phi_{tt}\} = \{0_{2x1}\}$$

$$\{\gamma\} \equiv -(\Phi_q \dot{q})_q \dot{q} - 2[\Phi_q]_t \dot{q} - \{\Phi_{tt}\}$$

$$\{\gamma\}_{REV} = -\dot{\phi}_i^2 [A_i] \{s_i\}^p \quad \text{for body } i$$

$$\{\gamma\}_{REV} = \dot{\phi}_j^2 [A_j] \{s_j\}^p \quad \text{for body } j$$

$$\{\gamma\}_{REV} = \dot{\phi}_j^2 [A_j] \{s_j\}^p - \dot{\phi}_i^2 [A_i] \{s_i\}^p$$