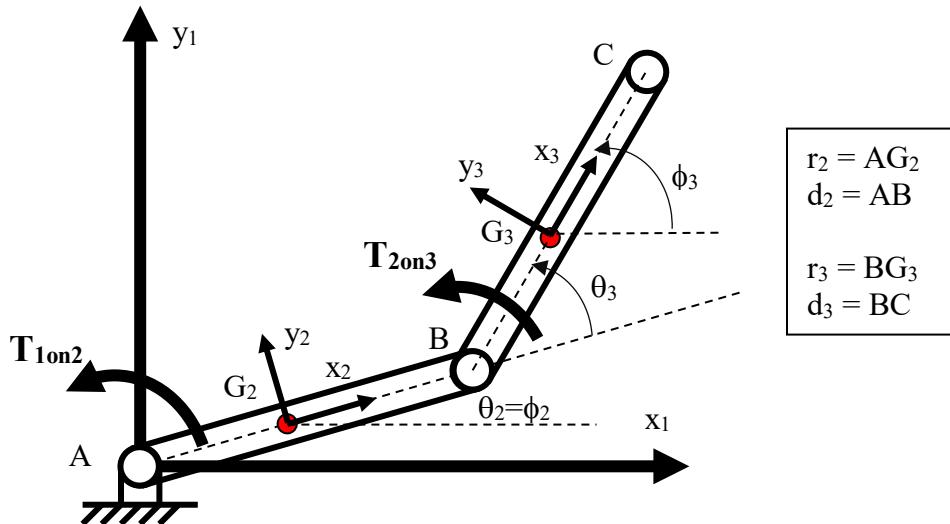


Differential-Algebraic Equations (DAE) for Anthropomorphic Manipulator



Two solid rigid bars with revolute joints A and B

Tool center point (TCP) at C (endpoint)

Centroids G_2 and G_3

Masses m_2 and m_3

Centroidal mass moments of inertia J_{G_2} and J_{G_3}

$T_{1\text{on}2}$ is torque of ground on bar 2 about revolute A measured CCW positive

$T_{2\text{on}3}$ is torque of bar 2 on bar 3 about revolute B measured CCW positive

Gravity g acts along negative y axis

$$nL = 3 \quad nJ1 = 2 \quad m = 3 \quad (nL-1) - 2 \quad nJ1 = 2$$

Lagrangian method (from Notes_09_02)

$$\{q\} = \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} \quad \dot{\{q\}} = \begin{Bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \quad \{Q\} = \begin{Bmatrix} T_{1 \text{ on } 2} \\ T_{2 \text{ on } 3} \end{Bmatrix} \quad \text{Note: } \theta_3 \text{ measured relative to } \theta_2$$

$$J_B = m_3 a_3^2 + J_3$$

$$J_A = J_B + m_2 a_2^2 + m_3 d_2^2 + J_2 + 2m_3 d_2 a_3 \cos \theta_3$$

$$C = J_B + m_3 d_2 a_3 \cos \theta_3$$

$$D = m_3 d_2 a_3 \sin \theta_3$$

$$G_2 = (m_2 a_2 + m_3 d_2) g \cos \theta_2$$

$$G_3 = m_3 a_3 g \cos(\theta_2 + \theta_3)$$

inverse dynamics

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} + \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix}$$

know driver motion at any time t - find $\{q\}$ $\dot{\{q\}}$ $\ddot{\{q\}}$

compute driver torques $T_{1 \text{ on } 2}$ $T_{2 \text{ on } 3}$ from Lagrangian equations

compute joint forces $F_{1 \text{ on } 2}^x$ $F_{1 \text{ on } 2}^y$ $F_{2 \text{ on } 3}^x$ $F_{2 \text{ on } 3}^y$ from Newtonian equations

may arbitrarily choose any other time t

forward dynamics

$$\begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{bmatrix} J_A & C \\ C & J_B \end{bmatrix}^{-1} \left(\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} - \begin{Bmatrix} -D\dot{\theta}_3^2 - 2D\dot{\theta}_2\dot{\theta}_3 \\ +D\dot{\theta}_2^2 \end{Bmatrix} - \begin{Bmatrix} G_2 + G_3 \\ G_3 \end{Bmatrix} \right)$$

know current state $\{q\}$ and $\dot{\{q\}}$ at current t

compute $\dot{\{q\}}$ from Lagrangian equations

compute $F_{1 \text{ on } 2}^x$ $F_{1 \text{ on } 2}^y$ $F_{2 \text{ on } 3}^x$ $F_{2 \text{ on } 3}^y$ from Newtonian equations

must integrate $\ddot{\{q\}}$ to get new $\{q\}$ and $\dot{\{q\}}$ at the next time step

Newtonian method (from free body diagrams)

$$\begin{aligned}
 F_{1\text{on}2}^x - F_{2\text{on}3}^x &= m_2 \ddot{x}_2 \\
 F_{1\text{on}2}^y - F_{2\text{on}3}^y - m_2 g &= m_2 \ddot{y}_2 \\
 \{s_2\}^A \times \{F_{1\text{on}2}\} - \{s_2\}^B \times \{F_{2\text{on}3}\} + T_{1\text{on}2} - T_{2\text{on}3} &= J_{G2} \ddot{\phi}_2 \\
 F_{2\text{on}3}^x &= m_3 \ddot{x}_3 \\
 F_{2\text{on}3}^y - m_3 g &= m_3 \ddot{y}_3 \\
 \{s_3\}^B \times \{F_{2\text{on}3}\} + T_{2\text{on}3} J_{G3} \ddot{\phi}_3 &
 \end{aligned}$$

inverse dynamics

distribution?

$$\left[\begin{array}{cccccc} +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ -(\{s_2\}^A)^Y & +(\{s_2\}^A)^X & +1 & (\{s_2\}^B)^Y & -(\{s_2\}^B)^X & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -(\{s_3\}^B)^Y & +(\{s_3\}^B)^X & +1 \end{array} \right] \left[\begin{array}{c} F_{1\text{on}2}^x \\ F_{1\text{on}2}^y \\ T_{1\text{on}2} \\ F_{2\text{on}3}^x \\ F_{2\text{on}3}^y \\ T_{2\text{on}3} \end{array} \right]^T = \left[\begin{array}{c} m_2 \ddot{x}_2 \\ m_2 \ddot{y}_2 + m_2 g \\ J_{G2} \ddot{\phi}_2 \\ m_3 \ddot{x}_3 \\ m_3 \ddot{y}_3 + m_3 g \\ J_{G3} \ddot{\phi}_3 \end{array} \right]$$

know driver motion at any time t - find positions, velocities and accelerations

compute joint forces $\{F_{1\text{on}2}^x \quad F_{1\text{on}2}^y \quad F_{2\text{on}3}^x \quad F_{2\text{on}3}^y \quad T_{1\text{on}2} \quad T_{2\text{on}3}\}^T$

may arbitrarily choose any other time t

forward dynamics

know current positions and velocities at current t

very cumbersome to compute joint forces and accelerations simultaneously

must integrate accelerations to get new positions and velocities at the next time step

know motion

knowing reactions accelerations

DAE dynamics

$$\{q\} = \begin{Bmatrix} x_2 \\ y_2 \\ \phi_2 \\ x_3 \\ y_3 \\ \phi_3 \end{Bmatrix} \quad \begin{aligned} \{s_2\}^A &= \begin{Bmatrix} -AG_2 \\ 0 \end{Bmatrix} & \{s_2\}^B &= \begin{Bmatrix} BG_2 \\ 0 \end{Bmatrix} \\ \{s_3\}^B &= \begin{Bmatrix} -BG_3 \\ 0 \end{Bmatrix} & \{s_3\}^C &= \begin{Bmatrix} CG_3 \\ 0 \end{Bmatrix} \end{aligned}$$

$$\underbrace{\{\Phi\}_{\text{KINEMATIC}} = \begin{Bmatrix} \{r_2\}^A - \{r_1\}^A \\ \{r_3\}^B - \{r_2\}^B \end{Bmatrix}}_{\text{inverse dynamics}} \quad \underbrace{\{\Phi\}_{\text{DRIVERS}} = \begin{Bmatrix} f_1(\{q\}, t) \\ f_2(\{q\}, t) \end{Bmatrix}}_{\text{forward dynamics}}$$

inverse dynamics

$$\left[\begin{array}{cc} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{array} \right]_{12 \times 12} \left\{ \begin{array}{c} \ddot{q} \\ \lambda \end{array} \right\}_{12 \times 1} = \left\{ \begin{array}{c} Q_{\text{EXT}} \\ \gamma \end{array} \right\}_{12 \times 1}$$

$$\boxed{\{\Phi\} = \begin{Bmatrix} \{r_2\}^A - \{r_1\}^A \\ \{r_3\}^B - \{r_2\}^B \\ \{\Phi\}_{\text{DRIVERS}} \end{Bmatrix}} \quad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \\ T_{1 \text{ on } 2} \\ T_{2 \text{ on } 3} \end{Bmatrix}$$

know driver motion at any time t , find $\{q\} \quad \dot{q} \quad \gamma \quad \{Q\}_{\text{EXT}} \quad [\Phi_q]$

compute \ddot{q} and λ simultaneously

may arbitrarily choose any other time t

forward dynamics

$$\left[\begin{array}{cc} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{array} \right]_{10 \times 10} \left\{ \begin{array}{c} \ddot{q} \\ \lambda \end{array} \right\}_{10 \times 1} = \left\{ \begin{array}{c} Q_{\text{EXT}} \\ \gamma \end{array} \right\}_{10 \times 1}$$

$$\boxed{\{\Phi\} = \begin{Bmatrix} \{r_2\}^A - \{r_1\}^A \\ \{r_3\}^B - \{r_2\}^B \end{Bmatrix}} \quad \{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \end{Bmatrix}$$

know current state $\{q\}$ and \dot{q} at current t , find $\gamma \quad \{Q\}_{\text{EXT}} \quad [\Phi_q]$

compute \ddot{q} and λ simultaneously

must integrate \ddot{q} to get new $\{q\}$ and \dot{q} at the next time step

Inverse Dynamics – joint interpolated motion

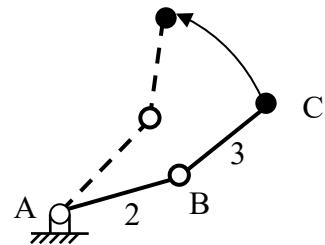
(independent position controllers on each joint)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \\ \phi_2 - \phi_{2_START} - \omega_2 t \\ \theta - \theta_{START} - \omega_\theta t \end{Bmatrix}$$

$$\omega_2 = (\phi_{2_END} - \phi_{2_START}) / \Delta t$$

$$\theta = \phi_3 - \phi_2$$

$$\omega_\theta = (\theta_{END} - \theta_{START}) / \Delta t$$



$$[\Phi_q]_{6x6} \quad \{v\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

$$\{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV_A} \\ \{\gamma\}_{REV_B} \\ 0 \\ 0 \end{Bmatrix}$$

$$\{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}$$

$$\{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \\ T_{1 \text{ on } 2} \\ T_{2 \text{ on } 3} \end{Bmatrix}$$

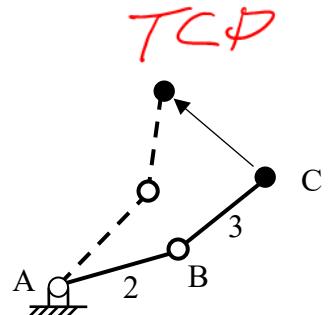
Inverse Dynamics – straight-line TCP interpolated motion

(interpolated position controllers on each joint)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \\ x_{C3} - x_{C3_START} - v_{C3_x} t \\ y_{C3} - y_{C3_START} - v_{C3_y} t \end{Bmatrix}$$

$$v_{C3_x} = (x_{C3_END} - x_{C3_START}) / \Delta t$$

$$v_{C3_y} = (y_{C3_END} - y_{C3_START}) / \Delta t$$



$$[\Phi_q]_{6x6} \quad \{v\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \\ v_{C3_X} \\ v_{C3_Y} \end{Bmatrix}$$

$$\{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV_A} \\ \{\gamma\}_{REV_B} \\ 0 \\ 0 \end{Bmatrix}$$

$$\{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}$$

$$\{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \\ F_{EFF_CX} \\ F_{EFF_CY} \end{Bmatrix}$$

$$\begin{aligned}\theta &= \phi_3 - \phi_2 \\ \dot{\theta} &= \dot{\phi}_3 - \dot{\phi}_2\end{aligned}\quad \left\{ \begin{aligned} \mathbf{r}_3 \end{aligned} \right\}^C = \begin{cases} r_2 \cos \phi_2 + r_3 \cos(\phi_2 + \theta) \\ r_2 \sin \phi_2 + r_3 \sin(\phi_2 + \theta) \end{cases}$$

$$\left\{ \dot{\mathbf{r}}_3 \right\}^C = \begin{bmatrix} -r_2 \sin \phi_2 - r_3 \sin(\phi_2 + \theta) & -r_3 \sin(\phi_2 + \theta) \\ r_2 \cos \phi_2 r_3 \cos(\phi_2 + \theta) & r_3 \cos(\phi_2 + \theta) \end{bmatrix} \begin{bmatrix} \dot{\phi}_2 \\ \dot{\theta} \end{bmatrix}$$

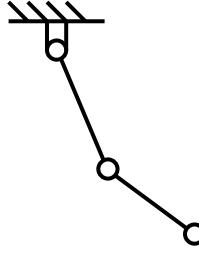
$$\boxed{\begin{bmatrix} F_{EFF_CX} \\ F_{EFF_CY} \end{bmatrix}^T \left\{ \dot{\mathbf{r}}_3 \right\}^C + \begin{bmatrix} T_{1on2} \\ T_{2on3} \end{bmatrix}^T \begin{bmatrix} \dot{\phi}_2 \\ \dot{\theta} \end{bmatrix} = 0}$$

$$\begin{bmatrix} T_{1on2} \\ T_{2on3} \end{bmatrix} = - \begin{bmatrix} -r_2 \sin \phi_2 - r_3 \sin(\phi_2 + \theta) & -r_3 \sin(\phi_2 + \theta) \\ r_2 \cos \phi_2 r_3 \cos(\phi_2 + \theta) & r_3 \cos(\phi_2 + \theta) \end{bmatrix}^T \begin{bmatrix} F_{EFF_CX} \\ F_{EFF_CY} \end{bmatrix}$$

Forward Dynamics – double pendulum

(no actuators)

$$\left\{ \Phi \right\} = \boxed{\begin{bmatrix} \left\{ \Phi \right\}_{REV_A} \\ \left\{ \Phi \right\}_{REV_B} \end{bmatrix}}$$



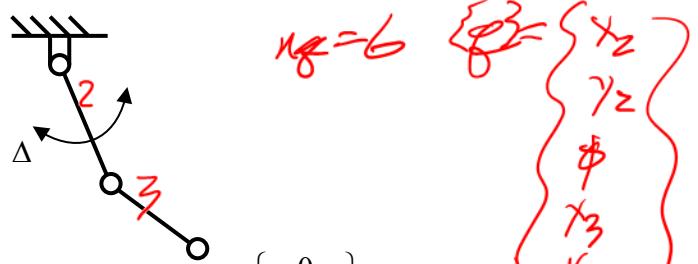
$$\begin{bmatrix} \Phi_q \\ 4x6 \end{bmatrix} \quad \left\{ v \right\} = \begin{bmatrix} \left\{ 0_{2x1} \right\} \\ \left\{ 0_{2x1} \right\} \end{bmatrix} \quad \left\{ \gamma \right\} = \begin{bmatrix} \left\{ \gamma \right\}_{REV_A} \\ \left\{ \gamma \right\}_{REV_B} \end{bmatrix} \quad \left\{ Q \right\}_{EXT} = \begin{bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{10x10} \begin{bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{bmatrix}_{10x1} = \begin{bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{bmatrix}_{10x1}}$$

$$\boxed{\left\{ \lambda \right\} = \begin{bmatrix} \left\{ F \right\}_{A1onA2} \\ \left\{ F \right\}_{B2onB3} \end{bmatrix}}$$

Forward Dynamics – proximal link kinematically driven, distal link pendulum
 (position controller only on proximal joint)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \\ \phi_2 - \phi_{2_CENTER} - \Delta \sin(2\pi f t) \end{Bmatrix}$$



$$[\Phi_q]_{5x6} \quad \{v\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \\ 2\pi f \Delta \cos(2\pi f t) \end{Bmatrix}$$

$$\{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV_A} \\ \{\gamma\}_{REV_B} \\ -4\pi^2 f^2 \Delta \sin(2\pi f t) \end{Bmatrix}$$

$$\{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ 0 \\ 0 \\ -m_3 g \\ 0 \end{Bmatrix}$$

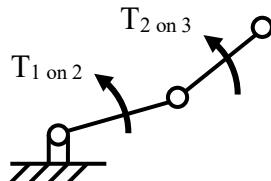
$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{11x11} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}_{11x1} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}_{11x1}$$

$$\{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \\ T_{1 \text{ on } 2} \end{Bmatrix}$$

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Forward Dynamics – computed torque control
 (torque controllers on each joint)

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{REV_A} \\ \{\Phi\}_{REV_B} \end{Bmatrix}$$



$$[\Phi_q]_{4x6} \quad \{v\} = \begin{Bmatrix} \{0_{2x1}\} \\ \{0_{2x1}\} \end{Bmatrix}$$

$$\{\gamma\} = \begin{Bmatrix} \{\gamma\}_{REV_A} \\ \{\gamma\}_{REV_B} \end{Bmatrix}$$

$$\{Q\}_{EXT} = \begin{Bmatrix} 0 \\ -m_2 g \\ T_{1 \text{ on } 2} - T_{2 \text{ on } 3} \\ 0 \\ -m_3 g \\ T_{2 \text{ on } 3} \end{Bmatrix}$$

$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}_{10x10} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix}_{10x1} = \begin{Bmatrix} \{Q\}_{EXT} \\ \{\gamma\} \end{Bmatrix}_{10x1}$$

$$\{\lambda\} = \begin{Bmatrix} \{F\}_{A1 \text{ on } A2} \\ \{F\}_{B2 \text{ on } B3} \end{Bmatrix}$$