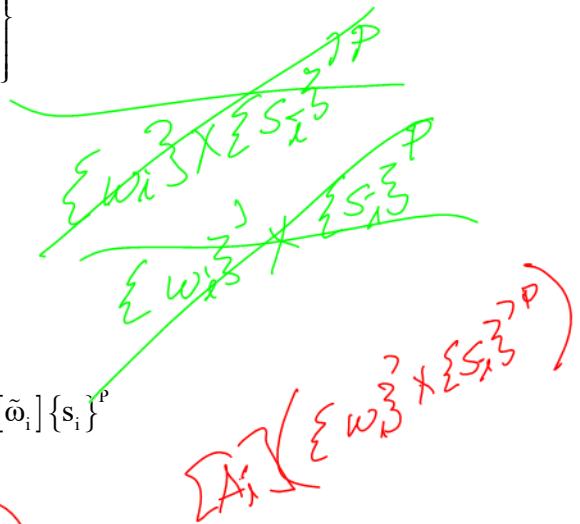


Three-Dimensional Kinematics

Position

$$\{\mathbf{r}_i\}^P = \{\mathbf{r}_i\} + [\mathbf{A}_i]\{\mathbf{s}_i\}'^P$$

$$\begin{bmatrix} \mathbf{x}_i \\ \mathbf{y}_i \\ \mathbf{z}_i \end{bmatrix}^P = \begin{bmatrix} \mathbf{x}_i \\ \mathbf{y}_i \\ \mathbf{z}_i \end{bmatrix} + \begin{bmatrix} \mathbf{a}_{i11} & \mathbf{a}_{i12} & \mathbf{a}_{i13} \\ \mathbf{a}_{i21} & \mathbf{a}_{i22} & \mathbf{a}_{i23} \\ \mathbf{a}_{i31} & \mathbf{a}_{i32} & \mathbf{a}_{i33} \end{bmatrix} \begin{bmatrix} \mathbf{x}_i \\ \mathbf{y}_i \\ \mathbf{z}_i \end{bmatrix}'^P$$



Velocity

$$\{\dot{\mathbf{r}}_i\}^P = \{\dot{\mathbf{r}}_i\} + [\dot{\mathbf{A}}_i]\{\mathbf{s}_i\}'^P$$

$$\{\dot{\mathbf{r}}_i\}^P = \{\dot{\mathbf{r}}_i\} + \{\boldsymbol{\omega}_i\} \times \{\mathbf{s}_i\}^P$$

$$\{\boldsymbol{\omega}_i\} = \begin{bmatrix} \omega_{ix} \\ \omega_{iy} \\ \omega_{iz} \end{bmatrix}$$

$$[\tilde{\boldsymbol{\omega}}_i] = \begin{bmatrix} 0 & -\omega_{iz} & \omega_{iy} \\ \omega_{iz} & 0 & -\omega_{ix} \\ -\omega_{iy} & \omega_{ix} & 0 \end{bmatrix}$$

$$\{\dot{\mathbf{r}}_i\}^P = \{\dot{\mathbf{r}}_i\} + [\tilde{\boldsymbol{\omega}}_i]\{\mathbf{s}_i\}^P = \{\dot{\mathbf{r}}_i\} + [\tilde{\boldsymbol{\omega}}_i][\mathbf{A}_i]\{\mathbf{s}_i\}'^P = \{\dot{\mathbf{r}}_i\} + [\mathbf{A}_i][\tilde{\boldsymbol{\omega}}_i]'\{\mathbf{s}_i\}'^P$$

$$[\dot{\mathbf{A}}_i] = [\tilde{\boldsymbol{\omega}}_i][\mathbf{A}_i] = [\mathbf{A}_i][\tilde{\boldsymbol{\omega}}_i]'$$

$$\{\mathbf{s}_i\}^P = [\mathbf{A}_i]\{\mathbf{s}_i\}'^P$$

$$\{\mathbf{s}_i\}'^P = [\mathbf{A}_i]^T \{\mathbf{s}_i\}^P$$

$$\{\boldsymbol{\omega}_i\} = [\mathbf{A}_i]\{\boldsymbol{\omega}_i\}'$$

$$\{\boldsymbol{\omega}_i\}' = [\mathbf{A}_i]^T \{\boldsymbol{\omega}_i\}$$

$$[\tilde{\boldsymbol{\omega}}_i] = [\mathbf{A}_i][\tilde{\boldsymbol{\omega}}_i]'[\mathbf{A}_i]^T$$

$$[\tilde{\boldsymbol{\omega}}_i]' = [\mathbf{A}_i]^T [\tilde{\boldsymbol{\omega}}_i][\mathbf{A}_i]$$

Acceleration

$$\{\ddot{\mathbf{r}}_i\}^P = \{\ddot{\mathbf{r}}_i\} + [\ddot{\mathbf{A}}_i]\{\mathbf{s}_i\}'^P$$

$$\{\ddot{\mathbf{r}}_i\}^P = \{\ddot{\mathbf{r}}_i\} + [\dot{\tilde{\boldsymbol{\omega}}}_i][\mathbf{A}_i]\{\mathbf{s}_i\}'^P + [\tilde{\boldsymbol{\omega}}_i][\dot{\mathbf{A}}_i]\{\mathbf{s}_i\}'^P = \{\ddot{\mathbf{r}}_i\} + [\dot{\tilde{\boldsymbol{\omega}}}_i][\mathbf{A}_i]\{\mathbf{s}_i\}'^P + [\tilde{\boldsymbol{\omega}}_i][\tilde{\boldsymbol{\omega}}_i][\mathbf{A}_i]\{\mathbf{s}_i\}'^P$$

$$\{\ddot{\mathbf{r}}_i\}^P = \{\ddot{\mathbf{r}}_i\} + [\beta_i][\mathbf{A}_i]\{\mathbf{s}_i\}'^P$$

$$[\beta_i] = [\dot{\tilde{\boldsymbol{\omega}}}_i] + [\tilde{\boldsymbol{\omega}}_i][\tilde{\boldsymbol{\omega}}_i]$$

$$[\ddot{\mathbf{A}}_i] = ([\dot{\tilde{\boldsymbol{\omega}}}_i] + [\tilde{\boldsymbol{\omega}}_i][\tilde{\boldsymbol{\omega}}_i])[\mathbf{A}_i] = [\beta_i][\mathbf{A}_i]$$

$$\ddot{\mathbf{A}}_i = \dot{\tilde{\boldsymbol{\omega}}}_i + \tilde{\boldsymbol{\omega}}_i \tilde{\boldsymbol{\omega}}_i$$

$$\{\ddot{r}_i\}^p = \{\ddot{r}_i\} + [\dot{A}_i][\tilde{\omega}_i]' \{s_i\}'^p + [A_i][\dot{\tilde{\omega}}_i]' \{s_i\}'^p = \{\ddot{r}_i\} + [A_i][\tilde{\omega}_i]'[\tilde{\omega}_i]' \{s_i\}'^p + [A_i][\dot{\tilde{\omega}}_i]' \{s_i\}'^p$$

$$\{\ddot{r}_i\}^p = \{\ddot{r}_i\} + [A_i][\beta_i]' \{s_i\}'^p \quad [\beta_i]' = [\dot{\tilde{\omega}}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'$$

$[\ddot{A}_i] = [A_i]\left([\dot{\tilde{\omega}}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'\right) = [A_i][\beta_i]'$

$$\{\dot{\omega}_i\} = [A_i]\{\dot{\omega}_i\}' \quad \{\dot{\omega}_i\}' = [A_i]^T \{\dot{\omega}_i\}$$

$$[\dot{\tilde{\omega}}_i] = [A_i][\dot{\tilde{\omega}}_i]'[A_i]^T \quad [\dot{\tilde{\omega}}_i]' = [A_i]^T[\dot{\tilde{\omega}}_i][A_i]$$

Jerk

$$\{\dddot{r}\}^p = \{\ddot{r}\} + [\ddot{A}_i]\{s_i\}'^p$$

$$\{\ddot{r}\}^p = \{\ddot{r}\} + \left([\ddot{\tilde{\omega}}_i] + 2[\dot{\tilde{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\dot{\tilde{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i] \right) [A_i]\{s_i\}'^p$$

$$[H_i] = 2[\dot{\tilde{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\dot{\tilde{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i] \quad \text{NOT angular momentum}$$

$$[H_i] = [A_i]\left([H_i]' + [\dot{\tilde{\omega}}_i]'[\tilde{\omega}_i]' - [\tilde{\omega}_i]'[\dot{\tilde{\omega}}_i]\right)[A_i]^T$$

$$\{\ddot{r}\}^p = \{\ddot{r}\} + \left([\ddot{\tilde{\omega}}_i] + [H_i] \right) [A_i]\{s_i\}'^p$$

$$[\ddot{A}_i] = \left([\ddot{\tilde{\omega}}_i] + 2[\dot{\tilde{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\dot{\tilde{\omega}}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\tilde{\omega}_i] \right) [A_i]$$

$$\{\ddot{r}\}^p = \{\ddot{r}\} + [A_i]\left([\ddot{\tilde{\omega}}_i]' + [\dot{\tilde{\omega}}_i]'[\tilde{\omega}_i]' + 2[\tilde{\omega}_i]'[\dot{\tilde{\omega}}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]' \right) \{s_i\}'^p$$

$$[H_i]' = 2[\tilde{\omega}_i]'[\dot{\tilde{\omega}}_i]' + [\dot{\tilde{\omega}}_i]'[\tilde{\omega}_i]' + [\tilde{\omega}_i]'[\tilde{\omega}_i]'[\tilde{\omega}_i]' \quad \text{NOT angular momentum}$$

$$[H_i]' = [A_i]^T \left([H_i] - [\dot{\tilde{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\dot{\tilde{\omega}}_i] \right) [A_i]$$

$$\{\ddot{\vec{r}}_i\}^p = \{\ddot{\vec{r}}_i\} + [A_i] \left([\ddot{\vec{\omega}}_i]' + [H_i]' \right) \{s_i\}'^p$$

$$[\ddot{\vec{A}}_i] = [A_i] \left([\ddot{\vec{\omega}}_i]' + [\dot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' + 2[\tilde{\vec{\omega}}_i]' [\dot{\vec{\omega}}_i]' + [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' \right)$$

$$\{\ddot{\vec{\omega}}_i\} = [A_i] ([\tilde{\vec{\omega}}_i]' \{ \dot{\vec{\omega}}_i \} + \{ \ddot{\vec{\omega}}_i \}') \quad \{ \ddot{\vec{\omega}}_i \}' = [A_i]^T (\{ \ddot{\vec{\omega}}_i \} - [\tilde{\vec{\omega}}_i] \{ \dot{\vec{\omega}}_i \})$$

$$[\ddot{\vec{\omega}}_i] = [A_i] \left([\ddot{\vec{\omega}}_i]' - [\dot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' + [\tilde{\vec{\omega}}_i]' [\dot{\vec{\omega}}_i]' \right) [A_i]^T \quad [\ddot{\vec{\omega}}_i]' = [A_i]^T \left([\ddot{\vec{\omega}}_i] + [\dot{\vec{\omega}}_i] [\tilde{\vec{\omega}}_i] - [\tilde{\vec{\omega}}_i] [\dot{\vec{\omega}}_i] \right) [A_i]$$

Snap

$$\{\ddot{\vec{r}}_i\}^p = \{\ddot{\vec{r}}_i\} + [\ddot{\vec{A}}_i] \{s_i\}'^p$$

$$\{\ddot{\vec{r}}_i\}^p = \left\{ \begin{array}{l} [\ddot{\vec{\omega}}_i] + 3[\ddot{\vec{\omega}}_i][\tilde{\vec{\omega}}_i] + 3[\dot{\vec{\omega}}_i][\dot{\vec{\omega}}_i] + [\tilde{\vec{\omega}}_i][\ddot{\vec{\omega}}_i] \\ + 3[\dot{\vec{\omega}}_i][\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i] + 2[\tilde{\vec{\omega}}_i][\dot{\vec{\omega}}_i][\tilde{\vec{\omega}}_i] + [\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i][\dot{\vec{\omega}}_i] \\ + [\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i] \end{array} \right\} [A_i] \{s_i\}'^p$$

$$[W_i] = 3[\ddot{\vec{\omega}}_i][\tilde{\vec{\omega}}_i] + 3[\dot{\vec{\omega}}_i][\dot{\vec{\omega}}_i] + [\tilde{\vec{\omega}}_i][\ddot{\vec{\omega}}_i] + 3[\dot{\vec{\omega}}_i][\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i] + 2[\tilde{\vec{\omega}}_i][\dot{\vec{\omega}}_i][\tilde{\vec{\omega}}_i] + [\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i][\dot{\vec{\omega}}_i] + [\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i]$$

$$[W_i] = [A_i] \left([W_i]' + 2[\ddot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' - 2[\tilde{\vec{\omega}}_i]' [\ddot{\vec{\omega}}_i] + 2[\dot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i] - 2[\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\dot{\vec{\omega}}_i] \right) [A_i]^T$$

$$\{\ddot{\vec{r}}_i\}^p = \{\ddot{\vec{r}}_i\} + ([\ddot{\vec{\omega}}_i] + [W_i]) [A_i] \{s_i\}'^p$$

$$\{\ddot{\vec{r}}_i\}^p = [A_i] \left\{ \begin{array}{l} [\ddot{\vec{\omega}}_i]' + [\ddot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' + 3[\dot{\vec{\omega}}_i]' [\dot{\vec{\omega}}_i] + 3[\tilde{\vec{\omega}}_i]' [\ddot{\vec{\omega}}_i]' \\ + [\dot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' + 2[\tilde{\vec{\omega}}_i]' [\dot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' + 3[\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\dot{\vec{\omega}}_i]' \\ + [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' \end{array} \right\} \{s_i\}'^p$$

$$[W_i]' = [\ddot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' + 3[\dot{\vec{\omega}}_i]' [\dot{\vec{\omega}}_i]' + 3[\tilde{\vec{\omega}}_i]' [\ddot{\vec{\omega}}_i]' + [\dot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' + 2[\tilde{\vec{\omega}}_i]' [\dot{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' + 3[\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\dot{\vec{\omega}}_i]' \\ + [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]' [\tilde{\vec{\omega}}_i]'$$

$$[W_i]' = [A_i]^T \left([W_i] - 2[\ddot{\vec{\omega}}_i][\tilde{\vec{\omega}}_i] + 2[\tilde{\vec{\omega}}_i][\ddot{\vec{\omega}}_i] - 2[\dot{\vec{\omega}}_i][\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i] + 2[\tilde{\vec{\omega}}_i][\tilde{\vec{\omega}}_i][\dot{\vec{\omega}}_i] \right) [A_i]$$

$$\{\ddot{r}_i\}^P = \{\ddot{r}_i\} + [A_i] \left([\ddot{\omega}_i]' + [W_i]' \right) \{s_i\}'^P$$

$$\{\ddot{\omega}_i\} = [A_i] \left(\{\ddot{\omega}_i\}' + 2[\tilde{\omega}_i]' \{\dot{\omega}_i\} + ([\dot{\tilde{\omega}}_i]' + [\tilde{\omega}_i]' [\tilde{\omega}_i]') \{\dot{\omega}_i\}' \right)$$

$$\{\ddot{\omega}_i\}' = [A_i]^T \left(\{\ddot{\omega}_i\} - 2[\tilde{\omega}_i] \{\dot{\omega}_i\} - ([\dot{\tilde{\omega}}_i] - [\tilde{\omega}_i][\tilde{\omega}_i]) \{\dot{\omega}_i\} \right)$$

$$[\ddot{\tilde{\omega}}_i] = [A_i] \begin{pmatrix} [\ddot{\tilde{\omega}}_i]' - 2[\tilde{\omega}_i]' [\tilde{\omega}_i] + 2[\tilde{\omega}_i]' [\ddot{\tilde{\omega}}_i]' \\ + [\dot{\tilde{\omega}}_i]' [\tilde{\omega}_i]' [\tilde{\omega}_i] - 2[\tilde{\omega}_i]' [\dot{\tilde{\omega}}_i]' [\tilde{\omega}_i] + [\tilde{\omega}_i]' [\tilde{\omega}_i]' [\dot{\tilde{\omega}}_i] \end{pmatrix} [A_i]^T$$

$$[\ddot{\tilde{\omega}}_i]' = [A_i]^T \begin{pmatrix} [\ddot{\tilde{\omega}}_i] + 2[\dot{\tilde{\omega}}_i][\tilde{\omega}_i] - 2[\tilde{\omega}_i][\ddot{\tilde{\omega}}_i] \\ + [\dot{\tilde{\omega}}_i][\tilde{\omega}_i][\tilde{\omega}_i] - 2[\tilde{\omega}_i][\dot{\tilde{\omega}}_i][\tilde{\omega}_i] + [\tilde{\omega}_i][\tilde{\omega}_i][\dot{\tilde{\omega}}_i] \end{pmatrix} [A_i]$$

2D partial derivatives

$$\{r_i\}^P = \{r_i\} + [A_i]\{s_i\}'^P$$

$$\text{by inspection } (\{r_i\}^P)_{ri} = [I_2] \quad (\{r_i\}^P)_{\phi i} = [B_i]\{s_i\}'^P$$

$$\{\dot{r}_i\}^P = \{\dot{r}_i\} + \dot{\phi}_i[B_i]\{s_i\}'^P$$

$$\{\dot{r}_i\}^P = ([I_2])\{\dot{r}_i\} + ([B_i]\{s_i\}'^P)\dot{\phi}_i + (0)t$$

$$\text{chain rule } \{\dot{r}_i\}^P = (\{r_i\}^P)_{ri}\{\dot{r}_i\} + (\{r_i\}^P)_{\phi i}\dot{\phi}_i + (\{r_i\}^P)_t$$

$$\text{compare to terms in chain rule } \underbrace{(\{r_i\}^P)_{ri} = [I_2]}_{\text{red line}} \quad \underbrace{(\{r_i\}^P)_{\phi i} = [B_i]\{s_i\}'^P}_{\text{red line}} \quad \underbrace{(\{r_i\}^P)_t = 0}_{\text{red line}}$$

3D partial derivatives

$$\{r_i\}^P = \{r_i\} + [A_i]\{s_i\}'^P$$

$$\{\dot{r}_i\}^P = \{\dot{r}_i\} + [A_i] \boxed{[\tilde{\omega}_i]' \{s_i\}'^P}$$

$$\{\dot{r}_i\}^P = \{\dot{r}_i\} + [A_i] (\{\omega_i\}' \times \{s_i\}'^P) = \{\dot{r}_i\} - [A_i] (\{s_i\}'^P \times \{\omega_i\}')$$

$$\{\dot{r}_i\}^p = \{\dot{r}_i\} - [A_i] \tilde{s}_i \cdot {}^p\{\omega_i\}'$$

$\{\omega_i\}' = 2[G_i] \dot{p}_i$

$$\{\dot{r}_i\}^p = \{\dot{r}_i\} - 2[A_i] \tilde{s}_i \cdot {}^p[G_i] \dot{p}_i$$

$$\{\dot{r}_i\}^p = ([I_3]) \{\dot{r}_i\} + (-2[A_i] \tilde{s}_i) \cdot {}^p[G_i] \dot{p}_i + (0)t$$

chain rule $\{\dot{r}_i\}^p = (\{\dot{r}_i\}^p)_{r_i} \{\dot{r}_i\} + (\{\dot{r}_i\}^p)_{p_i} \{\dot{p}_i\} + (\{\dot{r}_i\}^p)_t$

compare to terms in chain rule $(\{\dot{r}_i\}^p)_{r_i} = [I_3]$ $(\{\dot{r}_i\}^p)_{p_i} = -2[A_i] \tilde{s}_i \cdot {}^p[G_i]$ $(\{\dot{r}_i\}^p)_t = 0$

partial derivative wrt Euler parameters $(\{\dot{r}_i\}^p)_{p_i} = -2[A_i] \tilde{s}_i \cdot {}^p[G_i]$

$$\{\dot{r}_i\}^p = \{\dot{r}_i\} - [A_i] \tilde{s}_i \cdot {}^p\{\omega_i\}'$$

$$\{\dot{r}_i\}^p = ([I_3]) \{\dot{r}_i\} + (-[A_i] \tilde{s}_i) \cdot {}^p\{\omega_i\}' + (0)t$$

chain rule $\{\dot{r}_i\}^p = (\{\dot{r}_i\}^p)_{r_i} \{\dot{r}_i\} + (\{\dot{r}_i\}^p)_{\pi_i} \{\dot{\pi}_i\}' + (\{\dot{r}_i\}^p)_t$

compare to terms in chain rule $(\{\dot{r}_i\}^p)_{r_i} = [I_3]$ $(\{\dot{r}_i\}^p)_{\pi_i} = -[A_i] \tilde{s}_i$ $(\{\dot{r}_i\}^p)_t = 0$

partial derivative wrt π' directions $(\{\dot{r}_i\}^p)_{\pi_i} = -[A_i] \tilde{s}_i$

$$[*_{p_i}] = 2[*_{\pi_i}][G_i]$$

$$[*_{\pi_i}] = \frac{1}{2} [*_{p_i}] G_i^T$$

$$\frac{\partial}{\partial \{\dot{r}_i\}^p} \left(\begin{matrix} \{\dot{r}_i\}^p \\ \{\dot{\pi}_i\}^p \end{matrix} \right) = \begin{matrix} I_3 \\ 3 \times 3 \end{matrix}$$

$$\frac{\partial}{\partial \{\dot{r}_i\}^p} \left(\begin{matrix} \{\dot{r}_i\}^p \\ \{\dot{\pi}_i\}^p \end{matrix} \right) = \begin{matrix} I_4 \\ 3 \times 4 \end{matrix}$$