Three-Dimensional Dynamics

$[\Phi_r]\ \frac{\partial}{\partial\{r\}}$

$[\Phi_{\pi'}]\ \frac{\partial}{\partial\{\pi'\}}$

Must use centroidal coordinate frames

$$[\Phi_r] = \begin{bmatrix} [\Phi_{r2}]_{KINEMATIC} & [\Phi_{r3}]_{KINEMATIC} & \cdots \\ [\Phi_{r2}]_{DRIVER} & [\Phi_{r3}]_{DRIVER} & \cdots \end{bmatrix} \quad \text{for} \quad \{\ddot{r}\} = \begin{Bmatrix} \{\ddot{r}_2\} \\ \{\ddot{r}_3\} \\ \vdots \end{Bmatrix}$$

$\partial\{r_2\}$; $\partial\{r_3\}$; $\partial\{r_i\}$

$$[\Phi_{\pi'}] = \begin{bmatrix} [\Phi_{\pi'2}]_{KINEMATIC} & [\Phi_{\pi'3}]_{KINEMATIC} & \cdots \\ [\Phi_{\pi'2}]_{DRIVER} & [\Phi_{\pi'3}]_{DRIVER} & \cdots \end{bmatrix} \quad \text{for} \quad \{\omega\}' = \begin{Bmatrix} \{\omega_2\}' \\ \{\omega_3\}' \\ \vdots \end{Bmatrix} \quad \{\dot{\omega}\}' = \begin{Bmatrix} \{\dot{\omega}_2\}' \\ \{\dot{\omega}_3\}' \\ \vdots \end{Bmatrix}$$

$\partial\{\pi_2'\}$; $\partial\{\pi_3'\}$; $\partial\{\pi_i'\}$

$$[[\Phi_r] \quad [\Phi_{\pi'}]] \begin{Bmatrix} \{\dot{r}\} \\ \{\omega\}' \end{Bmatrix} = \{\nu\} \qquad [[\Phi_r] \quad [\Phi_{\pi'}]] \begin{Bmatrix} \{\ddot{r}\} \\ \{\dot{\omega}\}' \end{Bmatrix} = \{\gamma\}$$

$$[*_{pi}] = 2[*_{\pi' i}][G_i] \qquad \boxed{[*_{\pi' i}] = \tfrac{1}{2}[*_{pi}][G_i]^T}$$

$$[M_i] = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & m_i \end{bmatrix}$$

planar

$$[M_i] = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & J_{Gi} \end{bmatrix}$$

$[M]\{\ddot{q}\}$

$[J]'\{\dot{\omega}\}$

$[J_{Gi}]'$

all

$$\boxed{[M] = \begin{bmatrix} [M_2] & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & [M_3] & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & \ddots \end{bmatrix}}$$

9x9

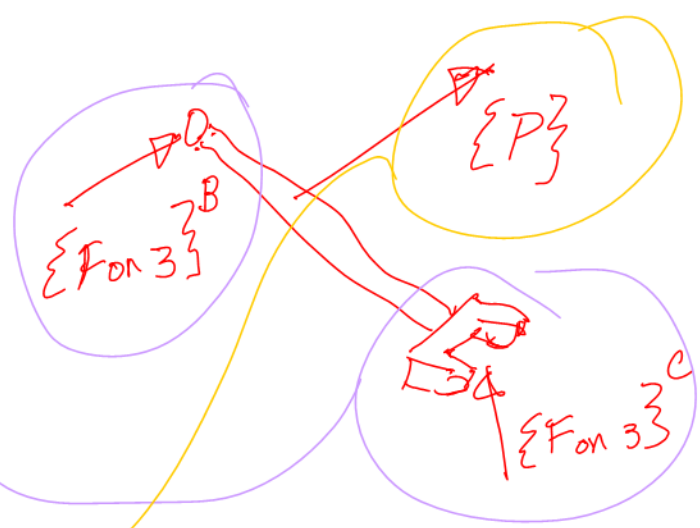
$$[J]' = \begin{bmatrix} [J_{G2}]' & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & [J_{G3}]' & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & \ddots \end{bmatrix}$$

9x9

$$[\tilde{\omega}]' = \begin{bmatrix} [\tilde{\omega}_2]' & 0_{3x3} & 0_{3x3} \\ 0_{3x3} & [\tilde{\omega}_3]' & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & \ddots \end{bmatrix}$$

9x9

$\{F_{on\,i}\}_{ALL}$ $\{F_{on\,i}\}_{APPLIED}$ $\{F_{on\,i}\}_{CONSTRAINT}$



$$\{F\}_{APPLIED} = \begin{Bmatrix} \Sigma\left(\{F_{on\ 2}\}_{APPLIED}\right) \\ \Sigma\left(\{F_{on\ 3}\}_{APPLIED}\right) \\ \vdots \end{Bmatrix}$$

only external forces

9x1

$$\{T\}'_{APPLIED} = \begin{Bmatrix} \Sigma\left(\{T_{on\ 2}\}'_{APPLIED}\right) \\ \Sigma\left(\{T_{on\ 3}\}'_{APPLIED}\right) \\ \vdots \end{Bmatrix}$$

only external torques

9x1

$[\Phi_r]^T$ mostly 0, +1, -1

$[\tilde{\pi}']^T$ moment arms

Equations of motion (EOM)

$$[M]\{\ddot{r}\} + [\Phi_r]^T\{\lambda\} = \{F\}_{APPLIED}$$

9x9 9x1 9x18 18x1 9x1

$$[J]'\{\dot{\omega}\}' + [\Phi_{\pi'}]^T\{\lambda\} = \{T\}'_{APPLIED} - [\tilde{\omega}]'[J]'\{\omega\}'$$

9x9 9x1 9x18 18x1 9x1 9x9 9x9 9x1

$$\begin{bmatrix} [\Phi_r] & [\Phi_{\pi'}] \end{bmatrix} \begin{Bmatrix} \{\ddot{r}\} \\ \{\dot{\omega}\}' \end{Bmatrix} = \{\gamma\} \qquad \text{or} \qquad \boxed{[\Phi_q]\{\ddot{q}\} = \{\gamma\}}$$

18x9 18x9 9x1 9x1 18x1 18x1 18x1

$\{\ddot{r}\}$ 3nb x 1	$[M]$ 3nb x 3nb	$\{F\}_{APPLIED}$ 3nb x 1	$[\Phi_r]$ nc x 3nb	$\{\lambda\}$ nc x 1
$\{\dot{\omega}\}'$ 3nb x 1	$[J]'$ 3nb x 3nb	$\{T\}'_{APPLIED}$ 3nb x 1	$[\tilde{\omega}]'[J]'\{\omega\}'$ 3nb x 1	$[\Phi_{\pi'}]$ nc x 3nb	$\{\gamma\}$ nc x 1

nb = number of moving bodies
nk = number of kinematic constraints
nd = number of driver constraints
nc = total number of constraints (nc = nk + nd)

$nc = nq$

$nc = nk + nd$

Inverse dynamics – kinematically driven

solve kinematics $\begin{Bmatrix} \{\ddot{r}\} \\ \{\dot{\omega}\}' \end{Bmatrix} = \begin{bmatrix} [\Phi_r] & [\Phi_{\pi'}] \end{bmatrix}^{-1}\{\gamma\}$

$\boxed{\begin{bmatrix} [\Phi_r] & [\Phi_{\pi'}] \end{bmatrix}}$ must have full rank nc = 6nb

$nc = nq$

$([\ ]^T)^{-1}$

compute constraint forces $\{\lambda\} = \left(\begin{bmatrix} [\Phi_r] & [\Phi_{\pi'}] \end{bmatrix}^T\right)^{-1} \begin{Bmatrix} \{F\}_{APPLIED} - [M]\{\ddot{r}\} \\ \{T\}'_{APPLIED} - [\tilde{\omega}]'[J]'\{\omega\}' - [J]'\{\dot{\omega}\}' \end{Bmatrix}$

$$\{\Phi\} = \begin{Bmatrix} \{\Phi\}_{KINEMATIC} \\ \{\Phi\}_{DRIVER} \end{Bmatrix} \qquad \{\lambda\} = \begin{Bmatrix} \{\lambda\}_{KINEMATIC} \\ \{\lambda\}_{DRIVER} \end{Bmatrix}$$

Statics

$$\{\ddot{r}\} = \{0\} \quad \text{and} \quad \{\omega\}' = \{0\} \quad \text{and} \quad \{\dot{\omega}\}' = \{0\}$$

$$\{\lambda\} = \left(\left[\left[\Phi_r\right] \quad \left[\Phi_{\pi'}\right] \right]^T \right)^{-1} \begin{Bmatrix} \{F\}_{APPLIED} \\ \{T\}'_{APPLIED} \end{Bmatrix}$$

Inverse dynamics – simultaneous EOM matrix

$$\begin{bmatrix} \underset{3nb \times 3nb}{[M]} & 0_{3nb \times 3nb} & \underset{3nb \times nc}{[\Phi_r]^T} \\ 0_{3nb \times 3nb} & \underset{3nb \times 3nb}{[J]'} & \underset{3nb \times nc}{[\Phi_{\pi'}]^T} \\ \underset{nc \times 3nb}{[\Phi_r]} & \underset{nc \times 3nb}{[\Phi_{\pi'}]} & 0_{nc \times nc} \end{bmatrix} \begin{Bmatrix} \underset{3nb \times 1}{\{\ddot{r}\}} \\ \underset{3nb \times 1}{\{\dot{\omega}\}'} \\ \underset{nc \times 1}{\{\lambda\}} \end{Bmatrix} = \begin{Bmatrix} \underset{3nb \times 1}{\{F\}_{APPLIED}} \\ \underset{3nb \times 1}{\{T\}'_{APPLIED} - [\tilde{\omega}]'[J]'\{\omega\}'} \\ \underset{nc \times 1}{\{\gamma\}} \end{Bmatrix}$$

list of external force
list of external torques

$$[EOM] = \underset{(6nb+nc) \times (6nb+nc)}{\begin{bmatrix} [M] & 0_{3nb \times 3nb} & [\Phi_r]^T \\ 0_{3nb \times 3nb} & [J]' & [\Phi_{\pi'}]^T \\ [\Phi_r] & [\Phi_{\pi'}] & 0_{nc \times nc} \end{bmatrix}}$$

$$\boxed{\begin{Bmatrix} \{\ddot{r}\} \\ \{\dot{\omega}\}' \\ \{\lambda\} \end{Bmatrix}} = \begin{bmatrix} [M] & 0_{3nb \times 3nb} & [\Phi_r]^T \\ 0_{3nb \times 3nb} & [J]' & [\Phi_{\pi'}]^T \\ [\Phi_r] & [\Phi_{\pi'}] & 0_{nc \times nc} \end{bmatrix}^{-1} \begin{Bmatrix} \{F\}_{APPLIED} \\ \{T\}'_{APPLIED} - [\tilde{\omega}]'[J]'\{\omega\}' \\ \{\gamma\} \end{Bmatrix}$$

RSUR
McPherson strut

36x36

Lagrange multipliers for specific constraints

$$\{F_{on\,i}\}_{CONSTRAINT} = -[\Phi_{r\,i}]^T_{CONSTRAINT}\{\lambda\}_{CONSTRAINT}$$

$$\{T_{on\,i}\}'_{CONSTRAINT} = \left([\tilde{s}_i]'^P[A_i]^T[\Phi_{r\,i}]^T_{CONSTRAINT} - [\Phi_{\pi'\,i}]^T_{CONSTRAINT}\right)\{\lambda\}_{CONSTRAINT}$$

$$\{F_{on\,j}\}_{CONSTRAINT} = -[\Phi_{r\,j}]^T_{CONSTRAINT}\{\lambda\}_{CONSTRAINT}$$

$$\{T_{on\,j}\}'_{CONSTRAINT} = \left([\tilde{s}_j]'^P[A_j]^T[\Phi_{r\,j}]^T_{CONSTRAINT} - [\Phi_{\pi'\,j}]^T_{CONSTRAINT}\right)\{\lambda\}_{CONSTRAINT}$$

local joint definition frame at P

$$\{s_i\}'^A = \{s_i\}'^P + [C_i]^P\{s_i\}''^A$$

$$\{F_{on\,i}\}'' = \left([C_i]^P\right)^T[A_i]^T\{F_{on\,i}\}$$

$$\{T_{on\,i}\}'' = \left([C_i]^P\right)^T\{T_{on\,i}\}'$$

Spherical

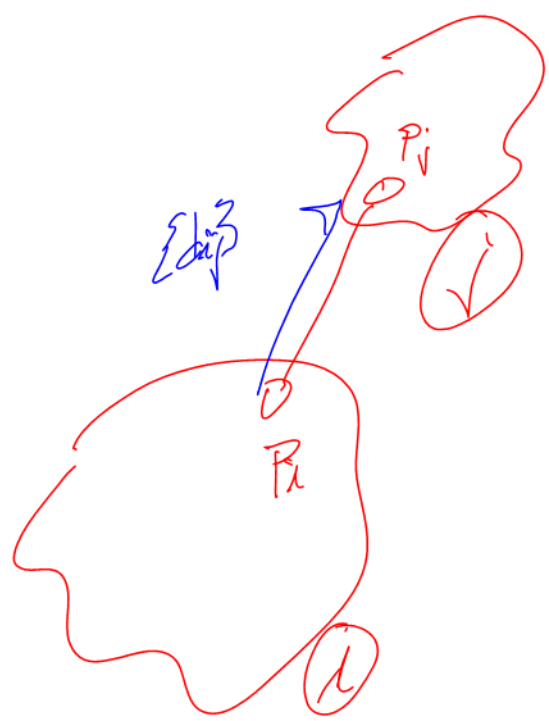
$$\{\Phi\}_{SPH} = \{r_j\}^P - \{r_i\}^P = \{0_{3x1}\}$$

$$\{F_{on\,i}\}_{SPH} = \{\lambda\}_{SPH}$$

$$\{T_{on\,i}\}'_{SPH} = \{0_{3x1}\}$$

$$\{F_{on\,j}\}_{SPH} = -\{\lambda\}_{SPH}$$

$$\{T_{on\,j}\}'_{SPH} = \{0_{3x1}\}$$



Double spherical

$$\Phi_{SPH_SPH} = \{d_{ij}\}^T\{d_{ij}\} - C^2 = 0 \qquad \text{for} \quad \{d_{ij}\} = \{r_j\}^P - \{r_i\}^P$$

$$\{F_{on\,i}\}_{CONSTRAINT} = 2\{d_{ij}\}\lambda_{SPH_SPH}$$

$$\lambda = \frac{F}{\text{norm}\{d_{ij}\}}$$

$$\{T_{on\ i}\}'_{CONSTRAINT} = \{0_{3x1}\}$$

$$\{F_{on\ j}\}_{CONSTRAINT} = -2\{d_{ij}\}\lambda_{SPH_SPH}$$

$$\{T_{on\ j}\}'_{CONSTRAINT} = \{0_{3x1}\}$$

Dot-1

$$\Phi_{DOT_1} = \{a_i\}^T\{a_j\} = 0 \qquad \text{for} \quad \{a_i\} = \{r_i\}^Q - \{r_i\}^P \quad \text{and} \quad \{a_j\} = \{r_j\}^Q - \{r_j\}^P$$

$$\{F_{on\ i}\}_{DOT_1} = \{0_{1x3}\}$$

$$\{T_{on\ i}\}'_{DOT_1} = -[\tilde{a}_i]'[A_i]^T[A_j]\{a_j\}'\lambda_{DOT_1}$$

$$\{F_{on\ j}\}_{DOT_1} = \{0_{1x3}\}$$

$$\{T_{on\ j}\}'_{DOT_1} = -[\tilde{a}_j]'[A_j]^T[A_i]\{a_i\}'\lambda_{DOT_1}$$

Dot-2

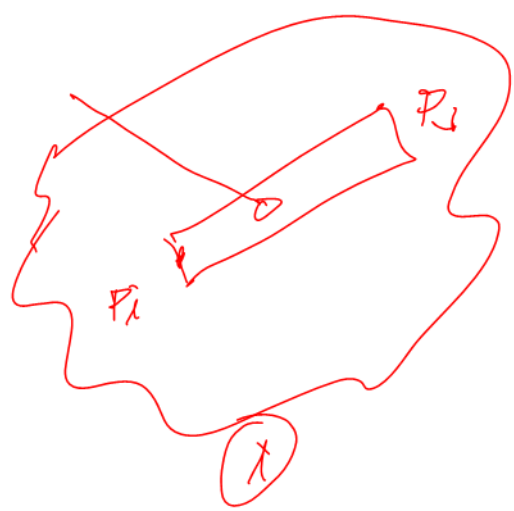
$$\Phi_{DOT_2} = \{a_i\}^T\{d_{ij}\} = 0 \qquad \text{for} \quad \{d_{ij}\} = \{r_j\}^P - \{r_i\}^P \quad \text{and} \quad \{a_i\} = \{r_i\}^Q - \{r_i\}^P$$

$$\{F_{on\ i}\}_{DOT_2} = \{a_i\}\lambda_{DOT_2}$$

$$\{T_{on\ i}\}'_{DOT_2} = -[\tilde{a}_i]'[A_i]^T\{d_{ij}\}\lambda_{DOT_2}$$

$$\{F_{on\ j}\}_{DOT_2} = -\{a_i\}\lambda_{DOT_2}$$

$$\{T_{on\ j}\}'_{DOT_2} = 0$$



Forward dynamics - dynamically driven

$$\left[\,[\Phi_r]\;\;[\Phi_{\pi'}]\,\right]\begin{Bmatrix}\{\ddot{r}\}\\ \{\dot{\omega}\}'\end{Bmatrix}=\{\gamma\}$$

nc < 6nb

$\boxed{\left[\,[\Phi_r]\;\;[\Phi_{\pi'}]\,\right]}$ does not have full row rank

given $\{r\}$ $\{p\}$ $\{\dot{r}\}$ $\{\omega\}'$ $\{F\}_{APPLIED}$ $\{T\}'_{APPLIED}$ $[M]$ $[J]'$

pos vel

compute $[\Phi_r]$ $[\Phi_{\pi'}]$ $\{\gamma\}$

$$\left[\,[\Phi_r]\;\;[\Phi_\phi]\,\right]\begin{Bmatrix}\{\ddot{r}\}\\ \{\ddot{\phi}\}\end{Bmatrix}=\{\gamma\}$$

$$\underbrace{\begin{Bmatrix}\{\ddot{r}\}\\ \{\dot{\omega}\}'\\ \{\lambda\}\end{Bmatrix}}_{35\times1}=\underbrace{\begin{bmatrix}[M] & 0_{3nb\times3nb} & [\Phi_r]^T\\ 0_{3nb\times3nb} & [J]' & [\Phi_{\pi'}]^T\\ [\Phi_r] & [\Phi_{\pi'}] & 0_{nc\times nc}\end{bmatrix}^{-1}}_{35\times35}\begin{Bmatrix}\{F\}_{APPLIED}\\ \{T\}'_{APPLIED}-[\tilde{\omega}]'[J]'\{\omega\}'\\ \{\gamma\}\end{Bmatrix}$$

18x1 17x1 18x1 17x1

must use forward time integration of $\{\ddot{r}\}$ $\{\dot{\omega}\}'$ to get $\{r\}$ $\{p\}$ $\{\dot{r}\}$ $\{\omega\}'$ at next time step

$$\underset{13nb}{\{y\}}=\begin{Bmatrix}\underset{3nb}{\{r\}}\\ \underset{4nb}{\{p\}}\\ \underset{3nb}{\{\dot{r}\}}\\ \underset{3nb}{\{\omega\}'}\end{Bmatrix}\quad \underset{13nb}{\{\dot{y}\}}=\begin{Bmatrix}\underset{3nb}{\{\dot{r}\}}\\ \underset{4nb}{\{\dot{p}\}}\\ \underset{3nb}{\{\ddot{r}\}}\\ \underset{3nb}{\{\dot{\omega}\}'}\end{Bmatrix}$$

pos pos vel vel

$$\boxed{\underset{4nb}{\{\dot{p}\}}=\tfrac{1}{2}[G]^T\underset{3nb}{\{\omega\}'}}\qquad \underset{3nb\times4nb}{[G]}=\begin{bmatrix}[G_2] & 0_{3x4} & 0_{3x4}\\ 0_{3x4} & [G_3] & 0_{3x4}\\ 0_{3x4} & 0_{3x4} & \ddots\end{bmatrix}$$

EOM using Euler parameters

$$\begin{bmatrix} [M] & 0_{3nb\ x\ 4nb} & [\Phi_r]^T & 0_{3nb\ x\ nb} \\ 0_{4nb\ x\ 3nb} & 4[G]^T[J]'[G] & [\Phi_p]^T & ([\Phi_p]_E)^T \\ [\Phi_r] & [\Phi_p] & 0_{nc\ x\ nc} & 0_{nc\ x\ nb} \\ 0_{nb\ x\ 3nb} & [\Phi_p]_E & 0_{nb\ x\ nc} & 0_{nb\ x\ nb} \end{bmatrix} \begin{Bmatrix} \{\ddot{r}\} \\ \{\ddot{p}\} \\ \{\lambda\} \\ \{\lambda\}_E \end{Bmatrix} = \begin{Bmatrix} \{F\}_{APPLIED} \\ 2[G]^T\{T\}'_{APPLIED} + 8[\dot{G}]^T[J]'[\dot{G}]\{p\} \\ \{\gamma\} \\ \{\gamma\}_E \end{Bmatrix}$$

$$\{p\} = \begin{Bmatrix} \{p_2\} \\ \{p_3\} \\ \vdots \end{Bmatrix} \qquad [G] = \begin{bmatrix} [G_2] & 0_{3x4} & 0_{3x4} \\ 0_{3x4} & [G_3] & 0_{3x4} \\ 0_{3x4} & 0_{3x4} & \ddots \end{bmatrix}$$

$$[\Phi_p]_E = 2\begin{bmatrix} \{p_2\}^T & 0_{1x4} & 0_{1x4} \\ 0_{1x4} & \{p_2\}^T & 0_{1x4} \\ 0_{1x4} & 0_{1x4} & \ddots \end{bmatrix} \qquad \{\gamma\}_E = -2\begin{Bmatrix} \{\dot{p}_2\}^T\{\dot{p}_2\} \\ \{\dot{p}_3\}^T\{\dot{p}_3\} \\ \vdots \end{Bmatrix}$$

Euler parameter

$$\{\Phi\}_{EUL} = \{p_i\}^T\{p_i\} - 1 = 0$$

$$[\Phi_{ri}]_{EUL} = \{0_{1x3}\}$$

$$[\Phi_{\pi' i}]_{EUL} = \{p_i\}^T[G_i]^T = ([G_i]\{p_i\})^T = \{0_{1x3}\}$$

$$[\Phi_{pi}]_{EUL} = 2\{p_i\}^T$$

$$\{F_{on\ i}\}_{EUL} = \{0_{3x1}\}$$

$$\{T_{on\ i}\}'_{EUL} = \{0_{3x1}\}$$

$$\{F_{on\ j}\}_{EUL} = \{0_{3x1}\}$$

$$\{T_{on\ j}\}'_{EUL} = \{0_{3x1}\}$$