Third-Order Differential-Algebraic Equations for Improved Integration of Multibody Dynamics

H.J. Sommer III

Mechanical and Nuclear Engineering

The Pennsylvania State University

experimental
kinematics
lab

Second-Order Differential-Algebraic Equations (DAE)

$[M]\{\ddot{q}\} + [\Phi_q]^T\{\lambda\} = \{Q^A\}$

external forces

mass

accel

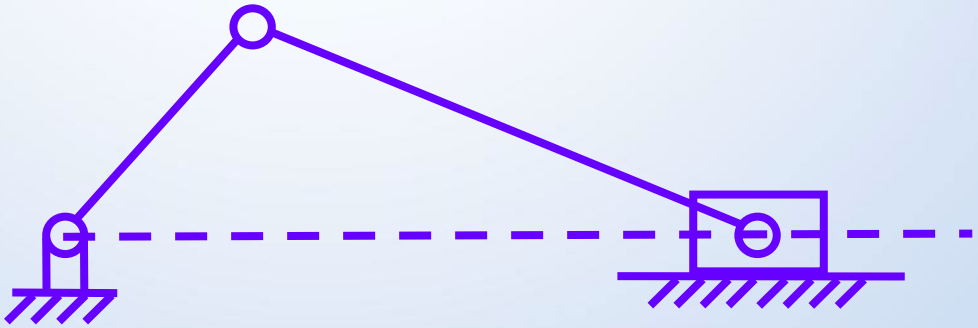
$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{Q^A\} \\ \{\gamma\} \end{Bmatrix}$$

Jacobian

joint forces

acc RHS

$[\Phi_q]\{\ddot{q}\} = \{\gamma\}$



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Forward dynamics

Second-Order DAE

Initial conditions and external forces

$$\{q\} \quad \{\dot{q}\} \quad \{Q^A\}$$

Solve DAE for accelerations

constants

ext

$$\begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}^{-1} \begin{Bmatrix} \{Q^A\} \\ \{\gamma\} \end{Bmatrix}$$

$\{q\}$ $\{q\}$ $\{\dot{q}\}$

Forward time integration

$$\{\ddot{q}\} \Rightarrow \{\dot{q}\} \Rightarrow \{q\}$$

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force slew rate variation

Simultaneous Third-Order DAE

mass — jerk — d/dt ext forces — FSR variation

$$\begin{bmatrix} [M] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix} \begin{Bmatrix} \{\dddot{q}\} \\ \{\dot{\lambda}\} \end{Bmatrix} = \begin{Bmatrix} \{\dot{Q}^A\} - \sum \{\kappa\} \\ \{\eta\} \end{Bmatrix}$$

Jacobian — d/dt jnt forces — jerk RHS

Obreshkov integration

$$\begin{Bmatrix} \{\ddot{q}\} \\ \{\dddot{q}\} \end{Bmatrix} \Rightarrow \{\dot{q}\} \Rightarrow \{q\}$$

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Kinematic Constraints

pos $$\mathbf{\Phi} = \mathbf{0}$$

$$\boxed{\mathbf{\Phi}_{\mathrm{q}}\dot{\mathbf{q}} = \mathbf{v}} \quad \text{for} \quad \mathbf{v} \equiv -\mathbf{\Phi}_{\mathrm{t}}$$

$$\boxed{\mathbf{\Phi}_{\mathrm{q}}\ddot{\mathbf{q}} = \boldsymbol{\gamma}} \quad \text{for} \quad \boldsymbol{\gamma} \equiv -\left(\mathbf{\Phi}_{\mathrm{q}}\dot{\mathbf{q}}\right)_{\mathrm{q}}\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathrm{q\,t}}\dot{\mathbf{q}} - \mathbf{\Phi}_{\mathrm{tt}}$$

$$\mathbf{\Phi}_{\mathrm{q}}\dddot{\mathbf{q}} = \boldsymbol{\eta} \quad \text{for}$$

$$\boldsymbol{\eta} \equiv -3\left(\mathbf{\Phi}_{\mathrm{q}}\dot{\mathbf{q}}\right)_{\mathrm{q}}\ddot{\mathbf{q}} - \left(\left(\mathbf{\Phi}_{\mathrm{q}}\dot{\mathbf{q}}\right)_{\mathrm{q}}\dot{\mathbf{q}}\right)_{\mathrm{q}}\dot{\mathbf{q}} - 3\mathbf{\Phi}_{\mathrm{q\,t}}\ddot{\mathbf{q}}$$

$$- 3\left(\mathbf{\Phi}_{\mathrm{q\,t}}\dot{\mathbf{q}}\right)_{\mathrm{q}}\dot{\mathbf{q}} - 3\mathbf{\Phi}_{\mathrm{q\,t\,t}}\dot{\mathbf{q}} - \mathbf{\Phi}_{\mathrm{ttt}}$$

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Jerk RHS for Planar Revolute

$$\mathbf{\Phi}^{\text{rev}} = \mathbf{d}_{\mathbf{ij}} = \mathbf{r}_{\mathbf{j}}^{\mathbf{P}} - \mathbf{r}_{\mathbf{i}}^{\mathbf{P}} = \mathbf{0}$$

$$\mathbf{\Phi}_{\text{qi}}^{\text{rev}} = \left[-\mathbf{I}\,, -\mathbf{B}_{\text{i}}\mathbf{s}_{\text{i}}{}'^{\text{P}} \right] \qquad \mathbf{\Phi}_{\text{qj}}^{\text{rev}} = \left[\mathbf{I}\,, \mathbf{B}_{\text{j}}\mathbf{s}_{\text{j}}{}'^{\text{P}} \right]$$

$$\boldsymbol{\nu}^{\text{rev}} = \mathbf{0}$$

$$\boldsymbol{\gamma}^{\text{rev}} = \mathbf{A}_{\text{j}}\mathbf{s}_{\text{j}}{}'^{\text{P}}\,\dot{\phi}_{\text{j}}^{2} - \mathbf{A}_{\text{i}}\mathbf{s}_{\text{i}}{}'^{\text{P}}\,\dot{\phi}_{\text{i}}^{2}$$

$$\boxed{\boldsymbol{\eta}^{\text{rev}} = 3\mathbf{A}_{\text{j}}\mathbf{s}_{\text{j}}{}'^{\text{P}}\,\dot{\phi}_{\text{j}}\ddot{\phi}_{\text{j}} + \mathbf{B}_{\text{j}}\mathbf{s}_{\text{j}}{}'^{\text{P}}\,\dot{\phi}_{\text{j}}^{3} - 3\mathbf{A}_{\text{i}}\mathbf{s}_{\text{i}}{}'^{\text{P}}\,\dot{\phi}_{\text{i}}\ddot{\phi}_{\text{i}} - \mathbf{B}_{\text{i}}\mathbf{s}_{\text{i}}{}'^{\text{P}}\,\dot{\phi}_{\text{i}}^{3}}$$

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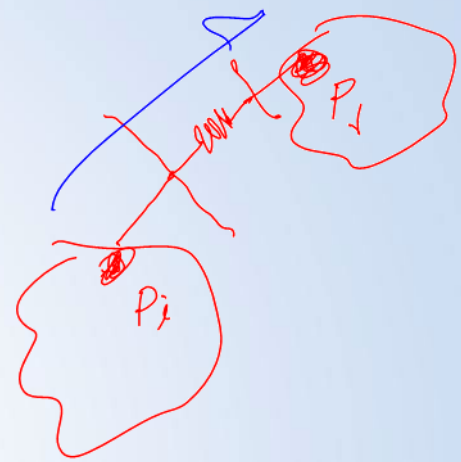
Also developed for planar double revolute, planar dot-1 and planar dot-2

Time Derivative of Actuator Forces

$$\mathbf{d_{ij}} = \mathbf{r_j^P} - \mathbf{r_i^P} \qquad d^2 = \mathbf{d_{ij}}^T \mathbf{d_{ij}}$$

$$\dot{d} = \mathbf{d_{ij}}^T \dot{\mathbf{d}}_{ij} / d \qquad \ddot{d} = \left(\mathbf{d_{ij}}^T \ddot{\mathbf{d}}_{ij} + \dot{\mathbf{d}}_{ij}^T \dot{\mathbf{d}}_{ij} - \dot{d}^2 \right) / d$$



$$f_{ACT} = k\left(d - d_0\right) + c\dot{d} + f\left(t, d, \dot{d}\right)$$

$$\mathbf{F}_{j_on_i} = f_{ACT}\, \mathbf{d_{ij}} / d \qquad \mathbf{F}_{i_on_j} = -\mathbf{F}_{j_on_i}$$

torque

$$n_{j_on_i} = \left(\mathbf{B_i}\, \mathbf{s_i}'^{\mathbf{P}} \right)^T \mathbf{F}_{j_on_i}$$

$$\frac{d}{dt}$$

$$\dot{f}_{ACT} = k\dot{d} + c\ddot{d} + \dot{f}\left(t, d, \dot{d}, \ddot{d}\right)$$

$$\dot{\mathbf{F}}_{j_on_i} = \left(\dot{f}_{ACT}\, \mathbf{d_{ij}}\, d + f_{ACT}\, \dot{\mathbf{d}}_{ij}\, d - f_{ACT}\, \mathbf{d_{ij}}\, \dot{d} \right) / d^2$$

$$\dot{n}_{j_on_i} = -\left(\dot{\varphi}_i \mathbf{A_i}\, \mathbf{s_i}'^{\mathbf{P}} \right)^T \mathbf{F}_{j_on_i} + \left(\mathbf{B_i}\, \mathbf{s_i}'^{\mathbf{P}} \right)^T \dot{\mathbf{F}}_{j_on_i}$$

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Planar Second-Order and Third-Order DAE

$$[M]\{\ddot{q}\} + [\Phi_q]^T\{\lambda\} = \{Q^A\}$$

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_\mathbf{q}^\mathbf{T} \\ \mathbf{\Phi}_\mathbf{q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}^\mathbf{A} \\ \boldsymbol{\gamma} \end{Bmatrix}$$

FSR variation

$$\boxed{\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_\mathbf{q}^\mathbf{T} \\ \mathbf{\Phi}_\mathbf{q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dddot{\mathbf{q}} \\ \dot{\boldsymbol{\lambda}} \end{Bmatrix} = \begin{Bmatrix} \dot{\mathbf{Q}}^\mathbf{A} - \left(\mathbf{\Phi}_\mathbf{q}^\mathbf{T}\boldsymbol{\lambda}\right)_\mathbf{q}\dot{\mathbf{q}} - \cancel{\mathbf{\Phi}_{\mathbf{q}\,\mathbf{t}}^\mathbf{T}\boldsymbol{\lambda}} \\ \boldsymbol{\eta} \end{Bmatrix}}$$

$$[M]\{\dddot{q}\} + \frac{d}{dt}\left([\Phi_q]^T\{\lambda\}\right) = \{\dot{Q}^A\}$$

$$[\Phi_q]^T\{\dot{\lambda}\} + \left([\Phi_q]^T\lambda\right)_q\{\dot{q}\} + \cancel{\frac{\partial}{\partial t}[\Phi_q]^T\{\lambda\}}$$

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Force Slew Rate (FSR) Variation

summation across all constraints k acting on body i

$$\left(\left(\Phi_q^T \lambda\right)_q \dot{q}\right)_{\text{BODY i}} = \sum_{k=1} \kappa_i^k$$

$$\kappa_i^k = \left(\left(\Phi_{qi}^k\right)^T \lambda^k\right)_{qi} \dot{q}_i + \left(\left(\Phi_{qi}^k\right)^T \lambda^k\right)_{qj} \dot{q}_j$$

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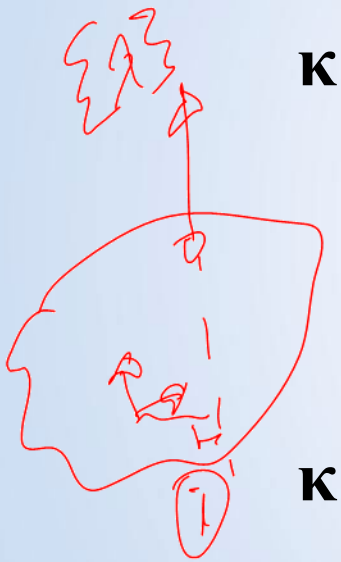
FSR Variation for Planar Revolute



$$\boldsymbol{\kappa}_{\mathbf{i}}^{\mathbf{rev}} = \left(\left(\left(\boldsymbol{\Phi}_{\mathbf{q}i}^{\mathrm{rev}} \right)^{\mathrm{T}} \boldsymbol{\lambda}^{\mathrm{rev}} \right)_{\mathbf{q}} \dot{\mathbf{q}} \right) = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ +\mathbf{s}_{\mathbf{i}}'^{\mathbf{P}\,\mathbf{T}} \mathbf{A}_{\mathbf{i}}^{\mathbf{T}} \boldsymbol{\lambda}^{\mathrm{rev}} \dot{\phi}_{\mathrm{i}} \end{Bmatrix}$$

$$\boldsymbol{\kappa}_{\mathbf{j}}^{\mathbf{rev}} = \left(\left(\left(\boldsymbol{\Phi}_{\mathbf{q}j}^{\mathrm{rev}} \right)^{\mathrm{T}} \boldsymbol{\lambda}^{\mathrm{rev}} \right)_{\mathbf{q}} \dot{\mathbf{q}} \right) = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\mathbf{s}_{\mathbf{j}}'^{\mathbf{P}\,\mathbf{T}} \mathbf{A}_{\mathbf{j}}^{\mathbf{T}} \boldsymbol{\lambda}^{\mathrm{rev}} \dot{\phi}_{\mathrm{j}} \end{Bmatrix}$$

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Also developed for planar double revolute, planar dot-1 and planar dot-2

Integrate Simultaneous DAE

Second-Order

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^{\mathbf{T}} \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}^{\mathbf{A}} \\ \boldsymbol{\gamma} \end{Bmatrix}$$

$$\dot{\mathbf{y}} = \mathbf{f}\left(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{Q}^{\mathbf{A}}, \mathrm{t}\right) = \begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix}$$

$$\{y\} = \begin{Bmatrix} \{q\} \\ \{f\} \end{Bmatrix}$$

Third-Order

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^{\mathbf{T}} \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dddot{\mathbf{q}} \\ \dot{\boldsymbol{\lambda}} \end{Bmatrix} = \begin{Bmatrix} \dot{\mathbf{Q}}^{\mathbf{A}} - \left(\mathbf{\Phi}_{\mathbf{q}}^{\mathbf{T}}\boldsymbol{\lambda}\right)_{\mathbf{q}} \dot{\mathbf{q}} - \mathbf{\Phi}_{\mathbf{q}\,\mathrm{t}}^{\mathbf{T}}\boldsymbol{\lambda} \\ \boldsymbol{\eta} \end{Bmatrix}$$

$$\ddot{\mathbf{y}} = \mathbf{g}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \mathbf{Q}^{\mathbf{A}}, \dot{\mathbf{Q}}^{\mathbf{A}}, \mathrm{t}\right) = \begin{Bmatrix} \ddot{\mathbf{q}} \\ \dddot{\mathbf{q}} \end{Bmatrix}$$

Same mass-Jacobian matrix

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Integrators

- Classical

E = Euler

$$y_{i+1} = y_i + f_i\, h$$

RK4 = Runge-Kutta

$$k_1 = f\left(y_i, t_i\right)$$
$$k_2 = f\left(y_i + \tfrac{1}{2} h\, k_1, t_i + \tfrac{1}{2} h\right)$$
$$k_3 = f\left(y_i + \tfrac{1}{2} h\, k_2, t_i + \tfrac{1}{2} h\right)$$
$$k_4 = f\left(y_i + h\, k_3, t_i + h\right)$$
$$y_{i+1} = y_i + \tfrac{1}{6} h\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

ABM4 = Adams-Bashforth-Moulton PC

$$y^*_{i+1} = y_i + h\left(23\, f_i - 16\, f_{i-1} + 5\, f_{i-2}\right)/12$$
$$y_{i+1} = y_i + h\left(9\, f^*_{i+1} + 19\, f_i - 5\, f_{i-1} + f_{i-2}\right)/24$$

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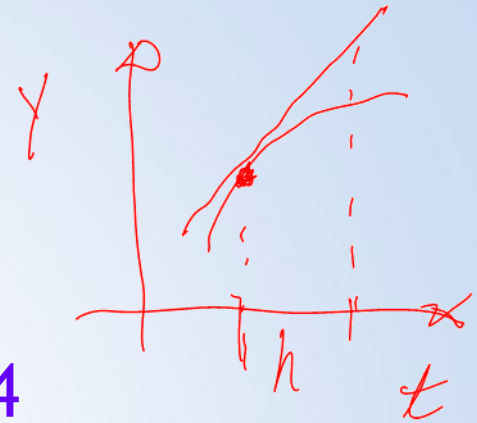
Integrators

- Second derivative



T = Taylor

$$y_{i+1} = y_i + f_i\, h + \tfrac{1}{2} g_i h^2$$

RKN4 = Runge-Kutta-Nystrom4

$$k_1 = g(y_i, t_i)$$
$$k_2 = g\left(y_i + \tfrac{1}{3} h f_i + \tfrac{1}{18} h^2 k_1, t_i + \tfrac{1}{3} h\right)$$
$$k_3 = g\left(y_i + \tfrac{2}{3} h f_i + \tfrac{2}{9} h^2 k_2, t_i + \tfrac{2}{3} h\right)$$
$$k_4 = g\left(y_i + h f_i + \tfrac{1}{6} h^2 (2k_1 + k_3), t_i + h\right)$$
$$y_{i+1} = y_i + h f_i + \tfrac{1}{120} h^2 (13k_1 + 36k_2 + 9k_3 + 2k_4)$$

OPC = Obreshkov Predictor-Corrector

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New Obreshkov Predictor-Corrector

$$y^*_{i+1} = y_i + h\left(-949\, f_i + 608\, f_{i-1} + 581\, f_{i-2}\right)/240$$
$$+ h^2\left(637\, g_i + 1080\, g_{i-1} + 173\, g_{i-2}\right)/240$$

$f \quad \frac{\partial y}{\partial x}$

$g \quad \frac{\partial^2 y}{\partial x^2}$

$$y_{i+1} = y_i + h\begin{pmatrix} 34465\, f^*_{i+1} + 42255\, f_i \\ +12015\, f_{i-1} + 1985\, f_{i-2} \end{pmatrix}/90720$$
$$+ h^2\begin{pmatrix} -3849\, g^*_{i+1} + 22977\, g_i \\ +7263\, g_{i-1} + 489\, g_{i-2} \end{pmatrix}/90720$$

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Integration Errors for DAE

- Solution error

  Explicit solution

  Reference solution with very small time step

- Position error

$$\{\Phi\} \neq 0$$

- Velocity error

$$\lfloor \Phi_q \rfloor \{\dot{q}\} \neq 0$$

- No constraint stabilization implemented

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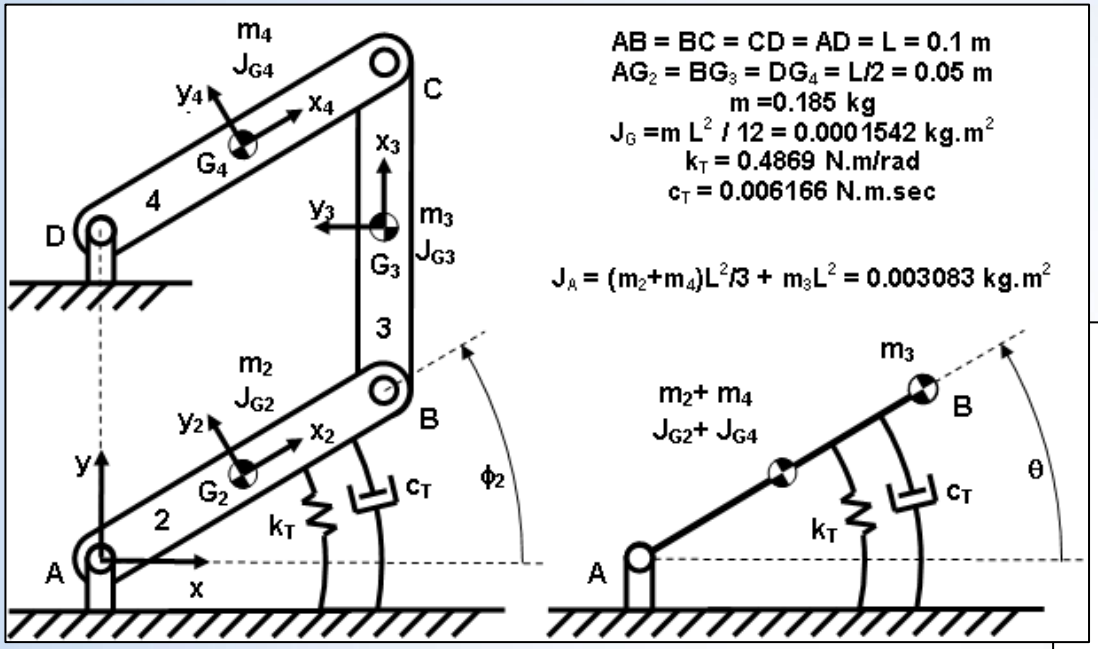
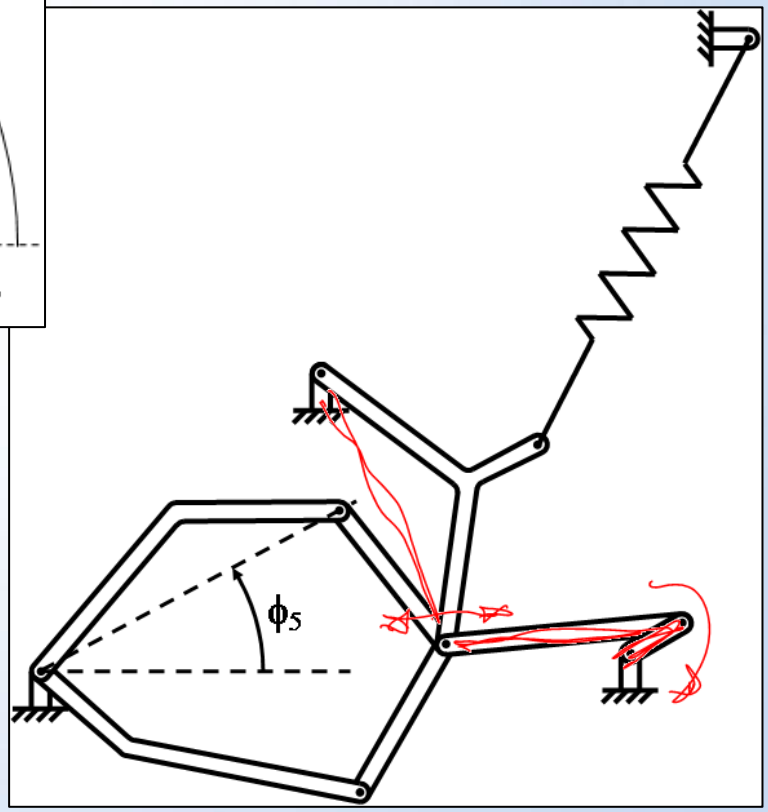
Planar Simulations





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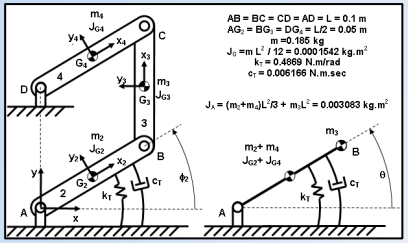
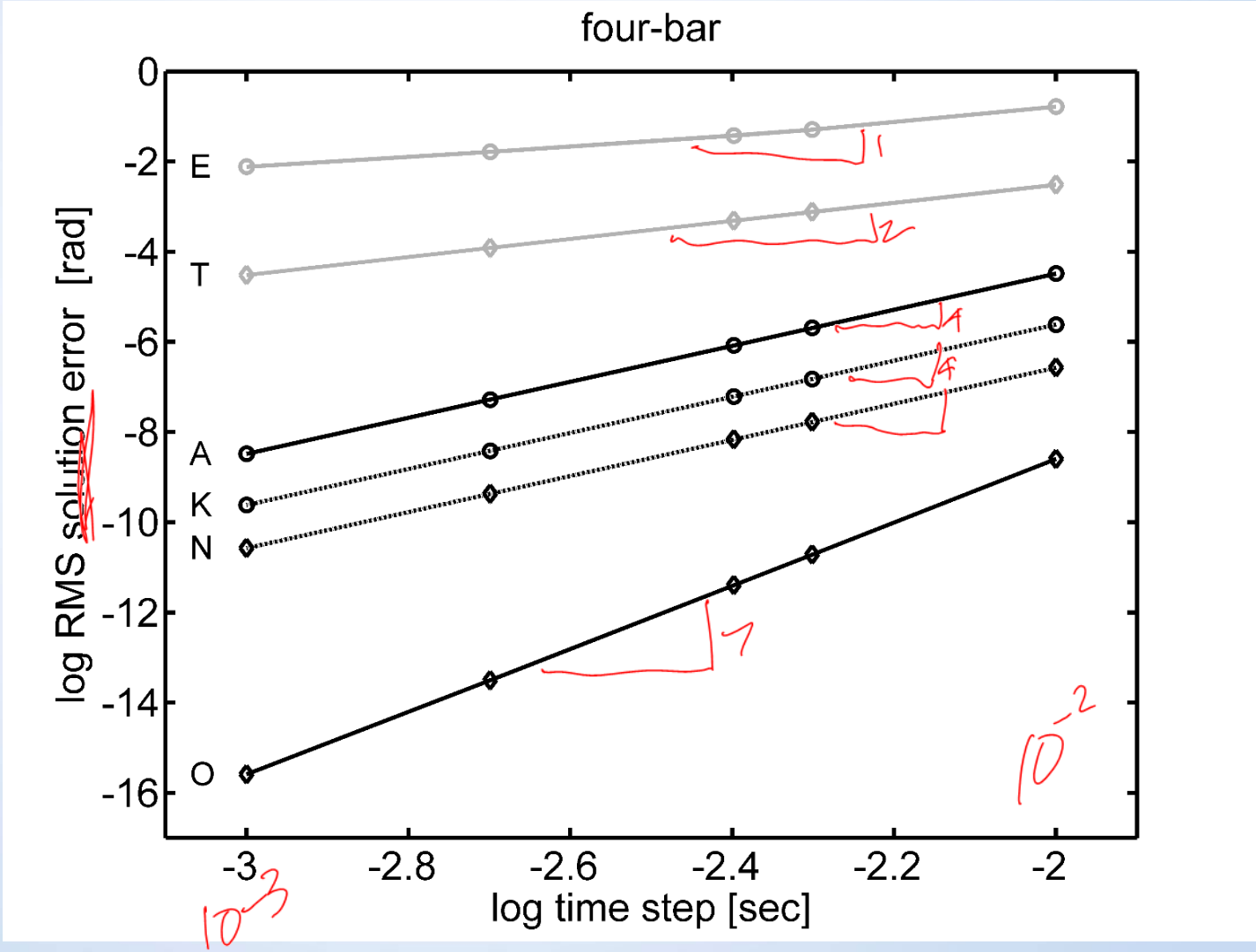
Four-bar Solution Error Versus Time Step



E = Euler
T = Taylor
A = ABM4
K = RK4
N = RKN4
O = OPC



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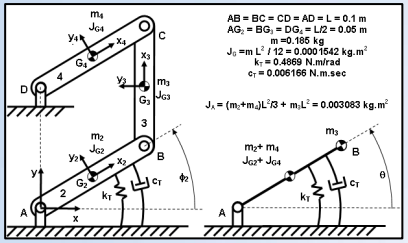
Four-bar Solution Error Versus CPU Time



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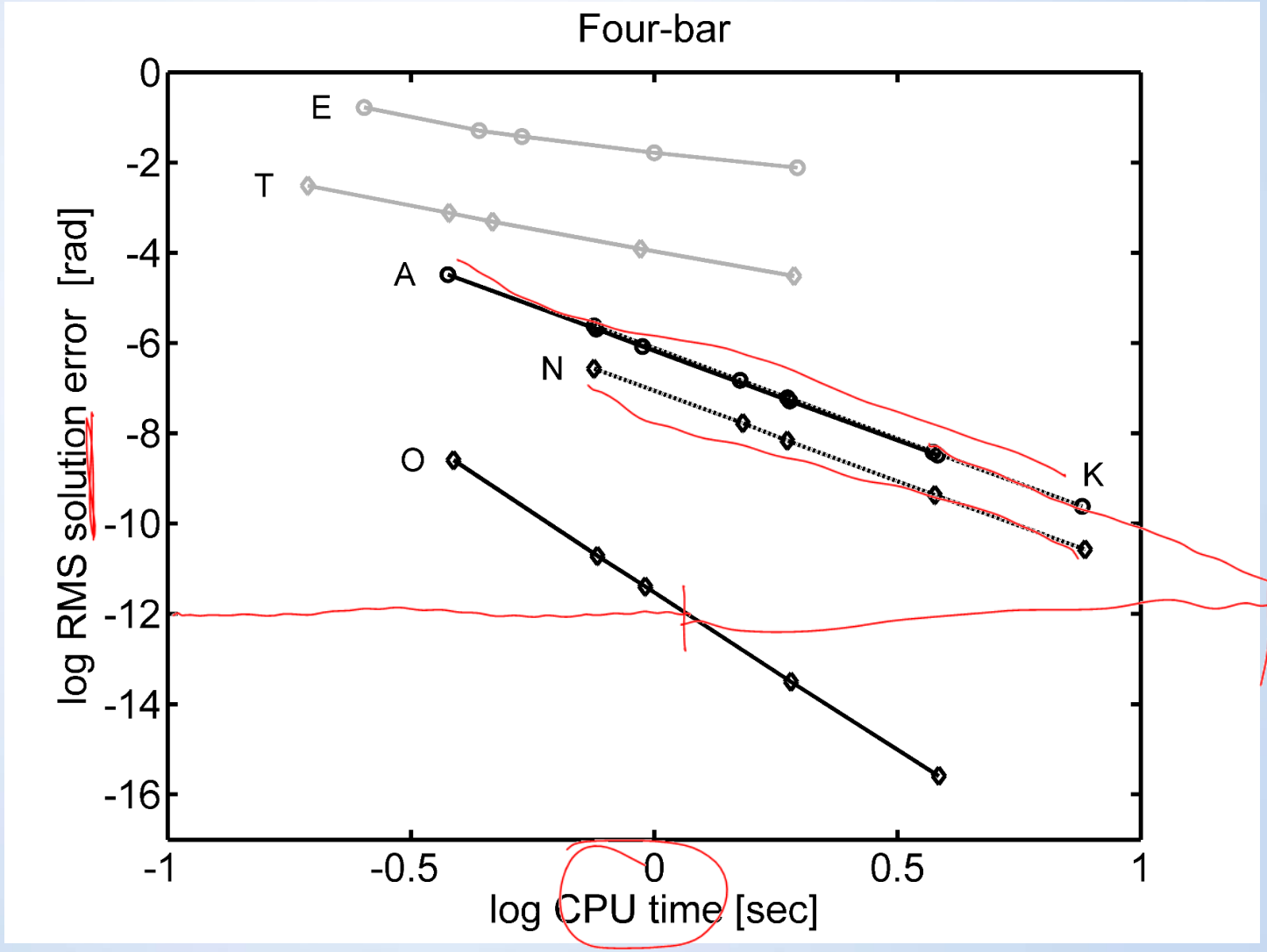

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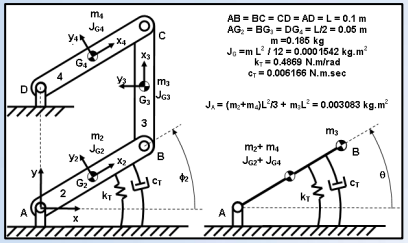
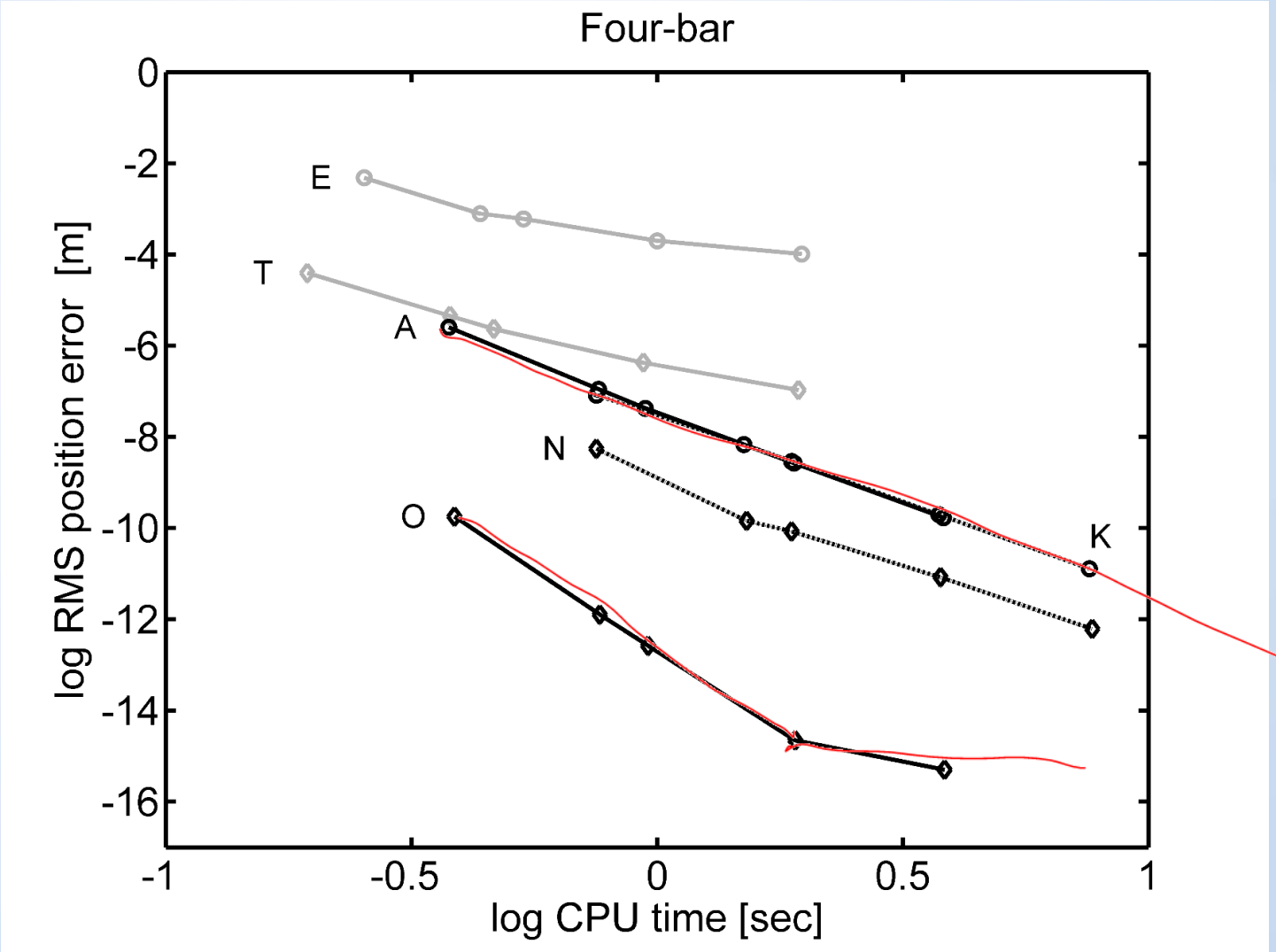
Four-bar Position Error Versus CPU Time



E = Euler
T = Taylor
A = ABM4
K = RK4
N = RKN4
O = OPC



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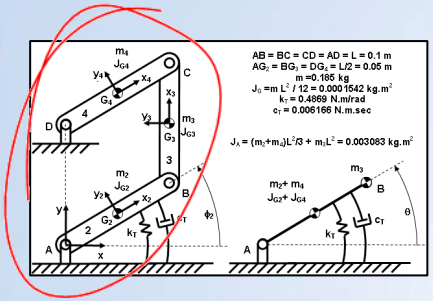
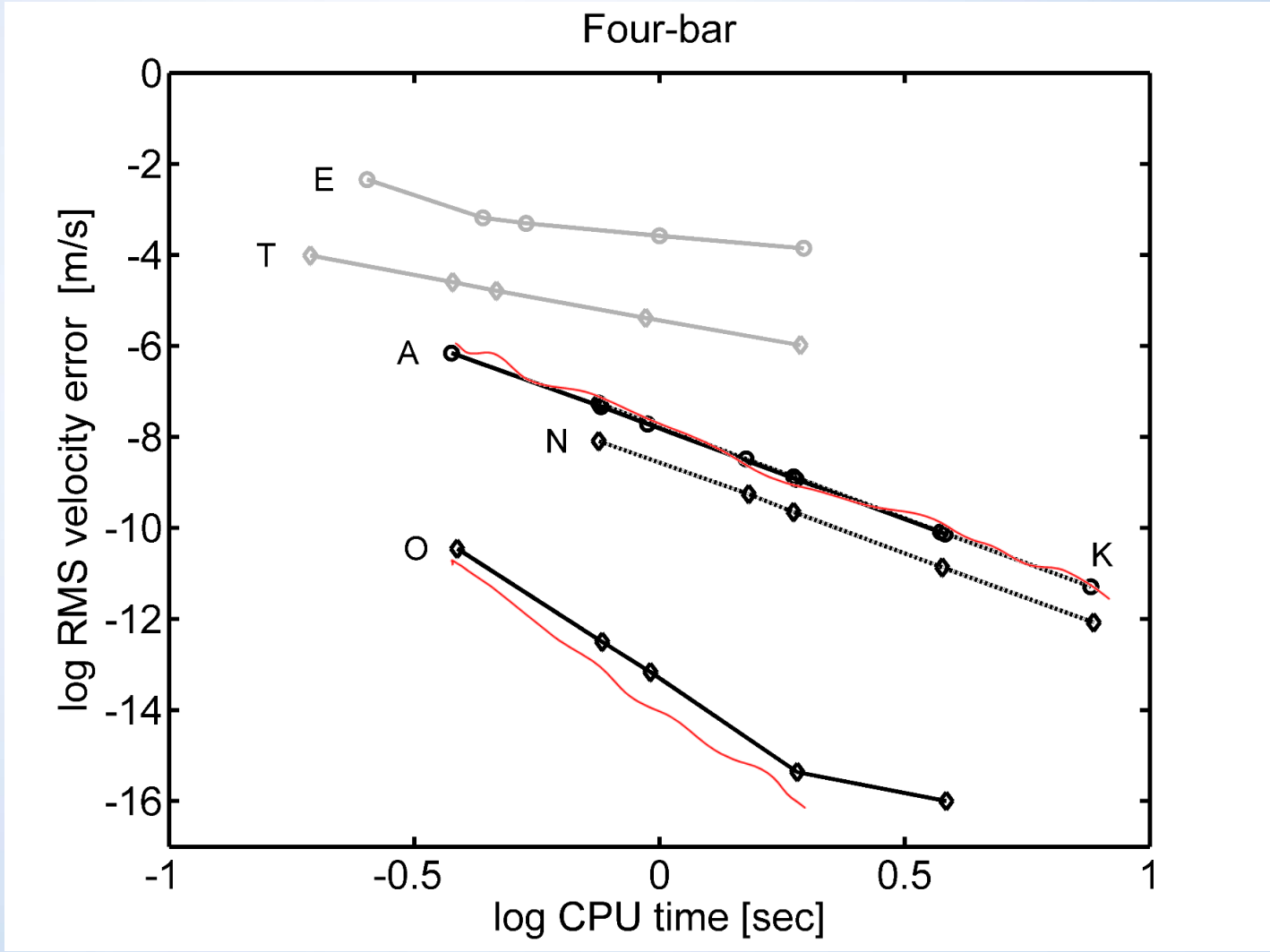
Four-bar Velocity Error Versus CPU Time



E = Euler
T = Taylor
A = ABM4
K = RK4
N = RKN4
O = OPC



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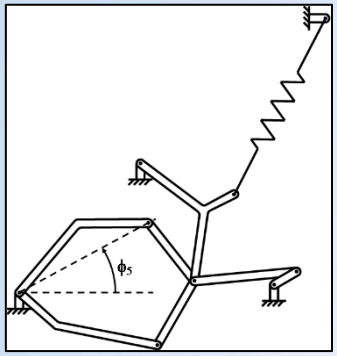
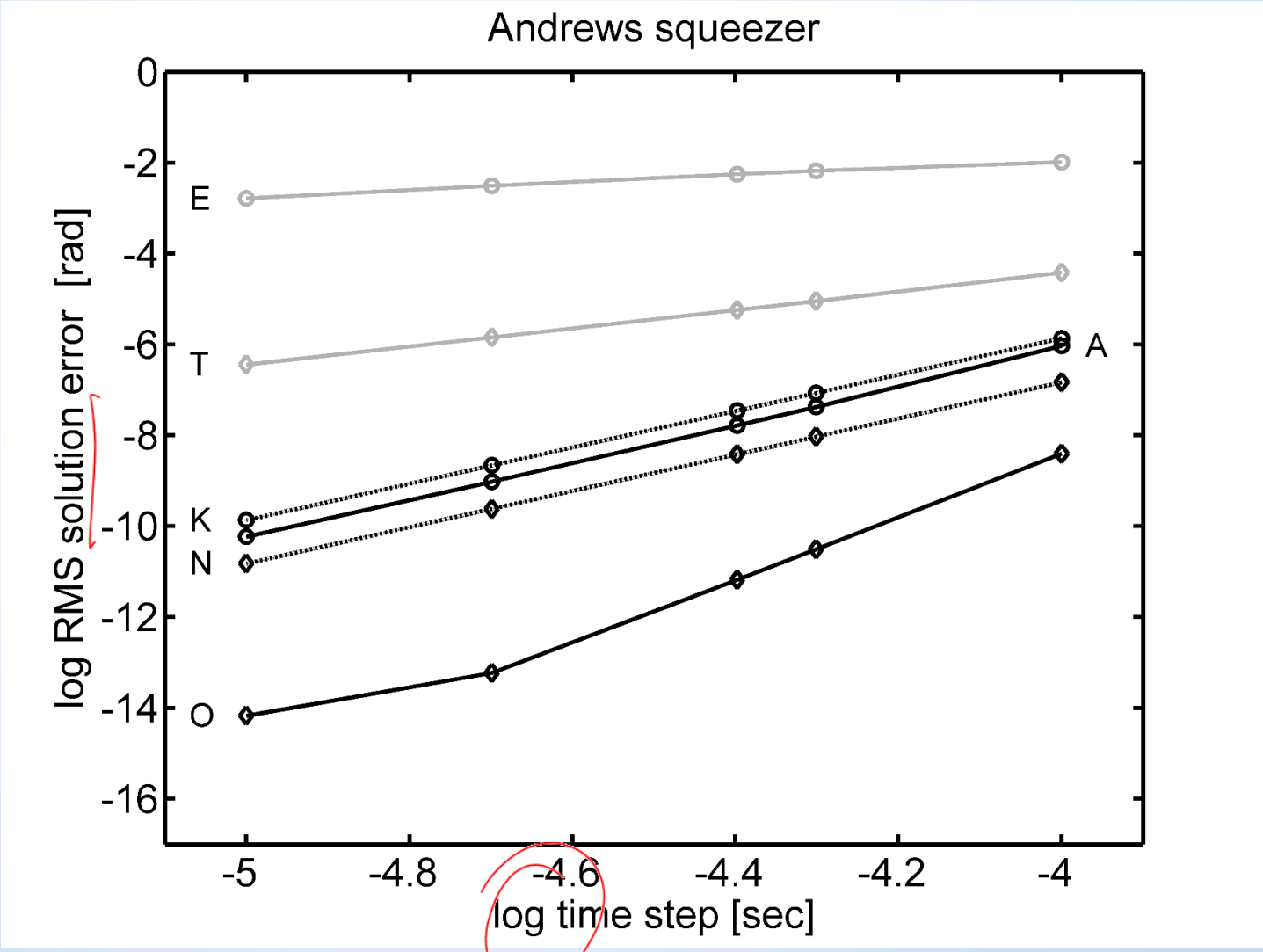
Andrews' Solution Error Versus Time Step



E = Euler
T = Taylor
A = ABM4
K = RK4
N = RKN4
O = OPC



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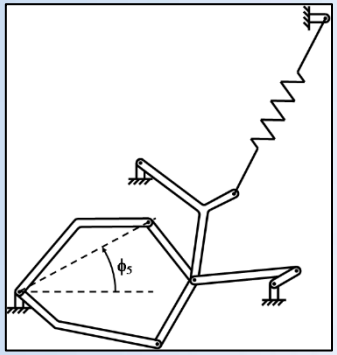
Andrews' Solution Error Versus CPU Time



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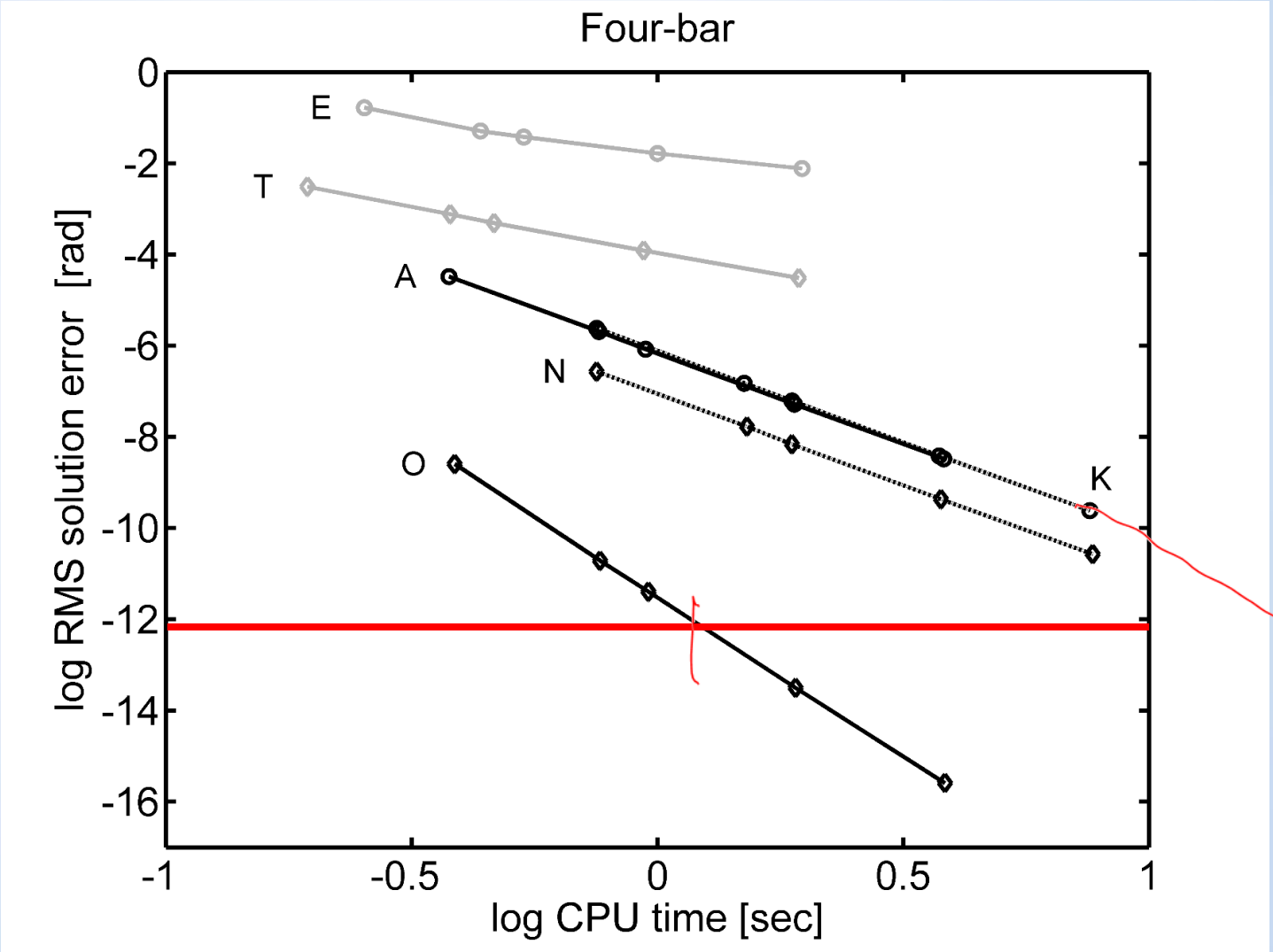

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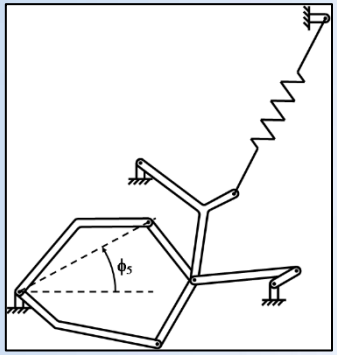
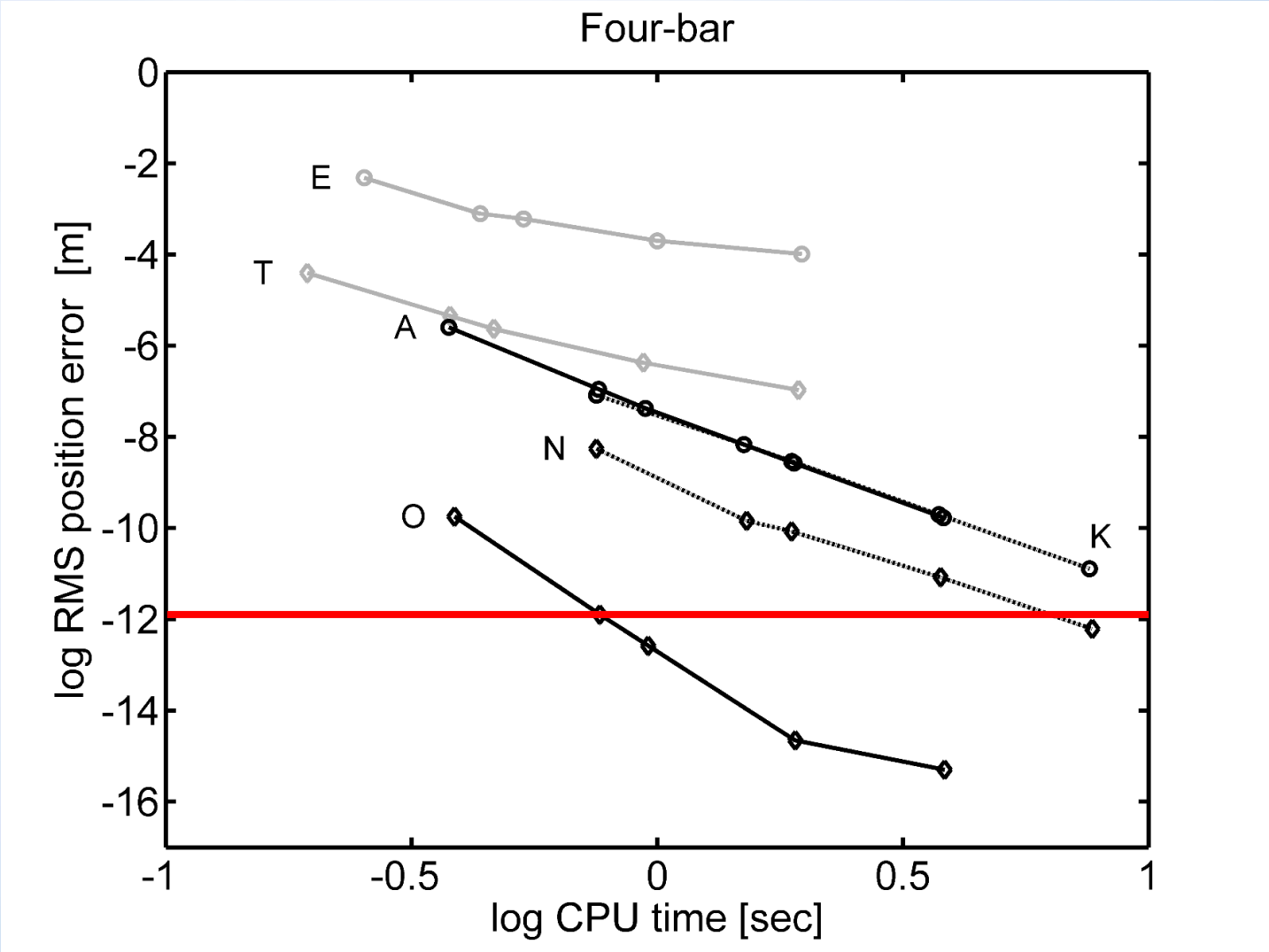
Andrews' Position Error Versus CPU Time



E = Euler
T = Taylor
A = ABM4
K = RK4
N = RKN4
O = OPC



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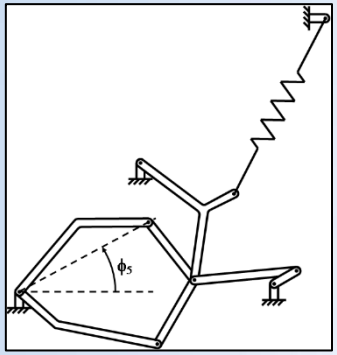
Andrews' Velocity Error Versus CPU Time



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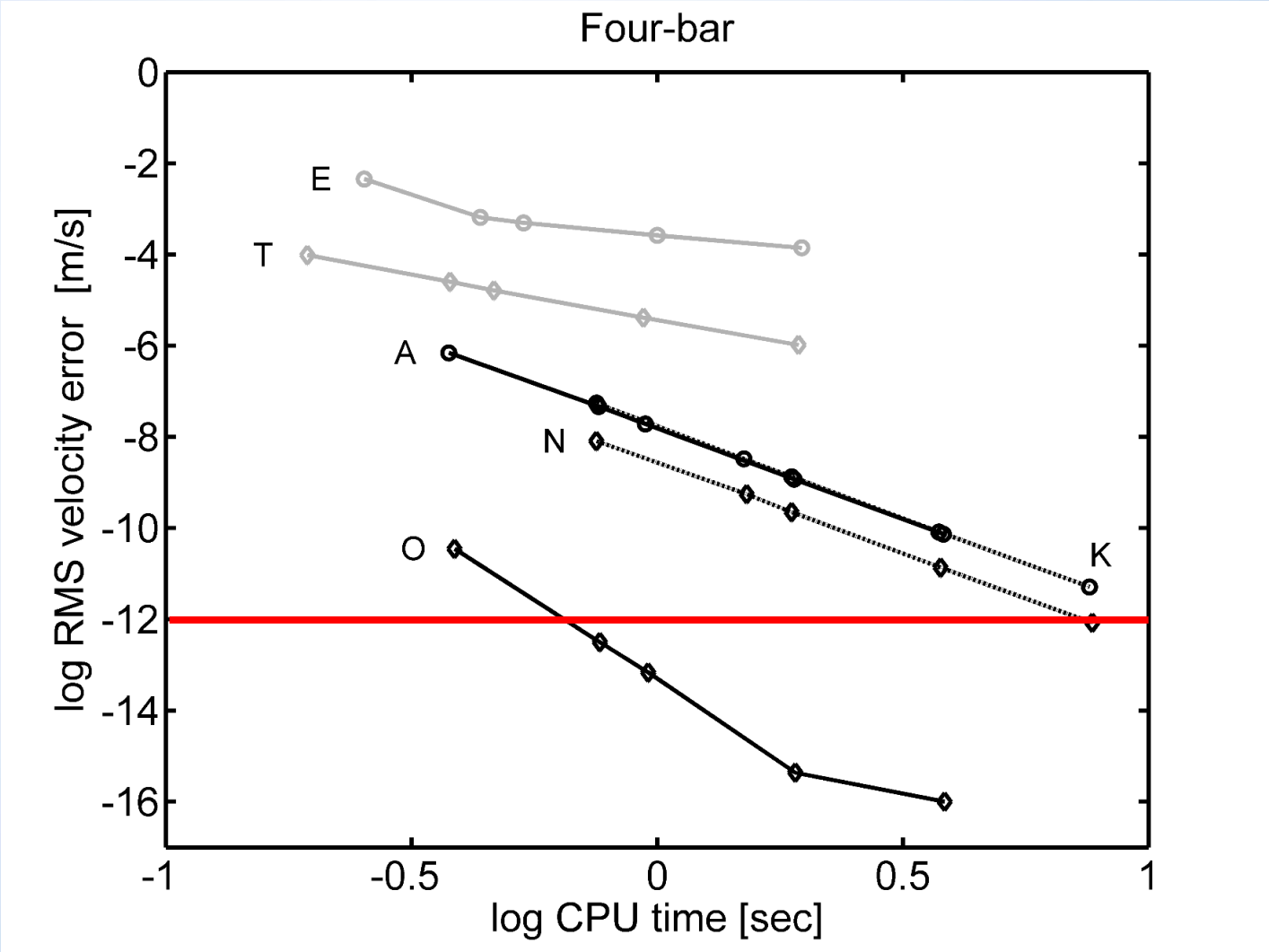
A = ABM4
K = RK4
N = RKN4
O = OPC



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Summary

- New jerk RHS terms for planar and 3D
- New third-order DAE for planar and 3D
- New FSR variations for planar and 3D
- New Obreskov predictor-corrector
- Simultaneous integration of second-order and third-order DAE is 10 times more accurate or conversely can be 10 times faster

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Future Work

- Implement Hilber-Hughes-Taylor
- Include constraint stabilization

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