(2) Solution of Gov. eq. (free vibration)

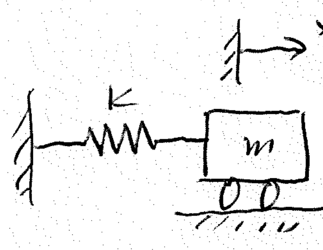
Math model in standard form:

$$a_2 \ddot{y} + a_0 y = 0 \quad (1) \qquad a_2, a_0 > 0$$

$y(t) = ?$

Need to know how and how much energy is stored in the system at $t = 0$

Or the initial conditions (ICs):

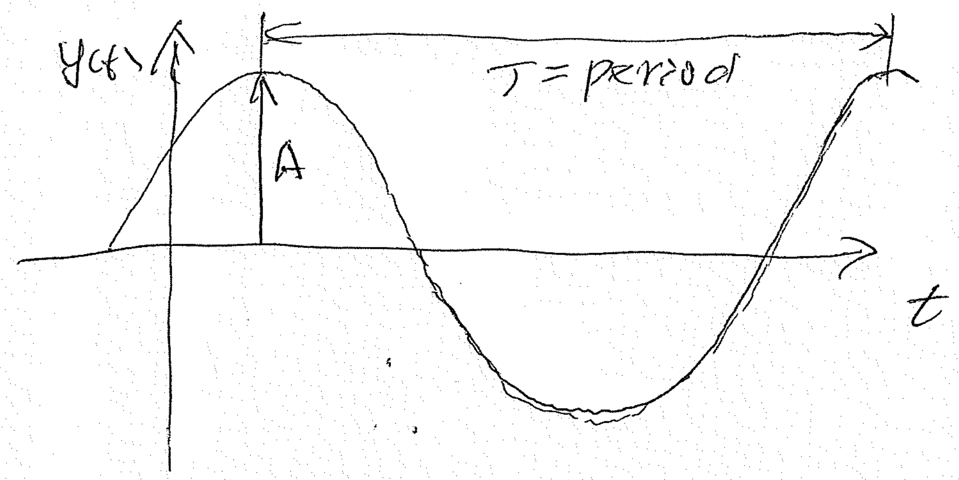


$y(0) = y_0$ —— PE stored

$\dot{y}(0) = \dot{y}_0$ —— KE stored

Propose a periodic solution:

$$y(t) = A \sin(\omega_n t + \phi) \quad (2)$$



A, ω_n and ϕ are constants.

A = amplitude of vibration

ω_n = frequency of vibration, $\omega_n T = 2\pi$

ϕ = phase shift

Is Eq. 2 correct?

It must satisfy Eq. 1 and the ICs.

Verify:

a) satisfying Eq. 1

Eq. (1) $\Rightarrow$

$$a_2 \frac{d^2}{dt^2}\left(A \sin(\omega_n t + \phi)\right) + a_0\left(A \sin(\omega_n t + \phi)\right) \stackrel{?}{=} 0$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(A \omega_n \cos(\omega_n t + \phi)\right)$$

$$= -A\omega_n^2 \sin(\omega_n t + \phi)$$

$$\Rightarrow \quad a_2\left[-A\omega_n^2 \sin(\omega_n t + \phi)\right] + a_0\left[A \sin(\omega_n t + \phi)\right] \stackrel{?}{=} 0$$

Or

$$-a_2\omega_n^2 + a_0 \stackrel{?}{=} 0 \qquad (3)$$

Yes, Eq. (2) satisfies Eq. (1) if and only if

ω_n in Eq. (2) is:

$$\boxed{\omega_n = \sqrt{\frac{a_0}{a_2}}} \qquad (4)$$

$$\boxed{\omega_n = \text{natural frequency of the system, } rad/s}$$

b) $y(t)$ must satisfy ICs

Eq. ② ⇒ $y(0) = \boxed{A \sin\phi = y_0}$ ⑤ ⇒ determine A and ϕ

and $\dot{y}(0) = \boxed{A\omega_n \cos\phi = \dot{y}_0}$ ⑥

yes, Eq. ② can satisfy ICs.

$A = ?$

$$\omega_n^2 (\text{Eq. (5)})^2 + (\text{Eq. ⑥})^2$$

$$\Rightarrow \omega_n^2 A^2 \sin^2\phi + A^2 \omega_n^2 \cos^2\phi = \boxed{\omega_n^2 A^2 = \omega_n^2 y_0^2 + \dot{y}_0^2}$$

$$\therefore \boxed{A = \left[y_0^2 + \frac{\dot{y}^2}{\omega_n^2}\right]^{1/2}} \quad ⑦$$

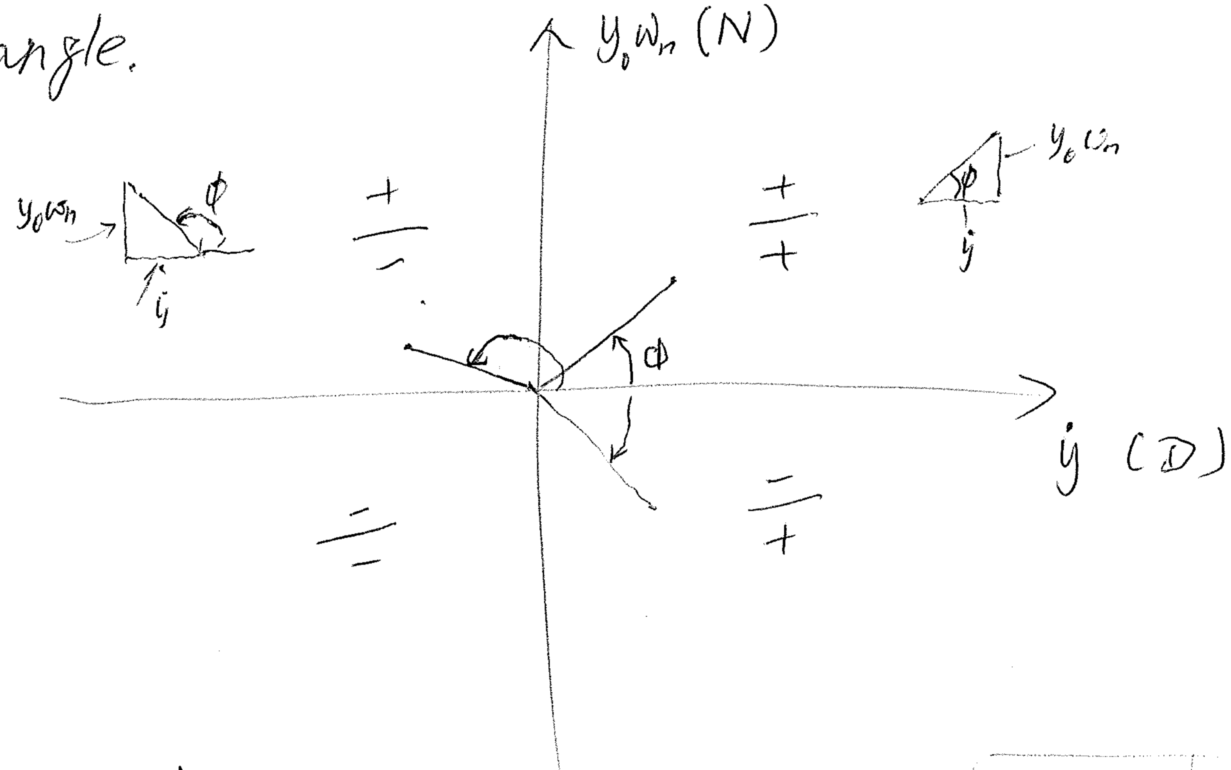
$\phi = ?$

$$\frac{\text{Eq ⑤}}{\text{Eq ⑥}} \Rightarrow \frac{\sin\phi}{\omega_n \cos\phi} = \frac{y_0}{\dot{y}_0} \Rightarrow \tan\phi = \frac{y_0 \omega_n}{\dot{y}_0}$$

$$\therefore \boxed{\phi = \tan^{-1} \frac{y_0 \omega_n}{\dot{y}_0}} \quad ⑧$$

~~But $\tan^{-1}$ function only defines angles in 1st and 4th quadrans~~

But $\tan^{-1}$ defines an angle in 1st or 4th quadran only while ϕ in 式(9) can be any angle.



So, define another $\tan^{-1}$ function: $\boxed{\tan_2^{-1}}$

$$\boxed{\phi = \tan_2^{-1} \frac{y_0\omega_n}{\dot{y}_0}} \equiv \begin{cases} \tan^{-1} \dfrac{y_0\omega_n}{\dot{y}_0} & \dot{y}_0 \geq 0 \\ \left(\tan^{-1} \dfrac{y_0\omega_n}{\dot{y}_0}\right) \pm \pi & \dot{y}_0 < 0 \end{cases} \quad (9)$$

Programing

$\phi = \tan_2^{-1} \dfrac{N}{D}$ → In Matlab: $\phi = \mathrm{atan2}(N, D)$

→ In excel: $\phi = \mathrm{atan2}(D, N)$

So, $\boxed{y(t) = A\sin(\omega_n t + \phi)}$ (2)

It may also be expressed in a different form.

Eq. (2) $\Rightarrow$ $y(t) = A\left(\sin\phi\cos\omega_n t + \cos\phi\sin\omega_n t\right)$

$$= \underbrace{(A\sin\phi)}_{A_1}\cos\omega_n t + \underbrace{(A\cos\phi)}_{A_2}\sin\omega_n t$$

$\Rightarrow$ $\boxed{y(t) = A_1\cos\omega_n t + A_2\sin\omega_n t}$ (2b)

$\dot y(0) = \left(-A_1\omega_n\sin\omega_n t + A_2\omega_n\cos\omega_n t\right)\Big|_{t=0}$

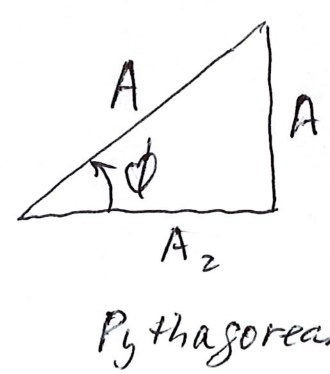
$y(0) = \boxed{A_1 = y_0}$

$\dot y(0) = A_2\omega_n = \dot y_0 \Rightarrow \boxed{A_2 = \frac{\dot y_0}{\omega_n}}$

$\therefore$

$$\boxed{A = \sqrt{A_1^2 + A_2^2} \qquad \phi = \tan^{-1}_{2}\frac{A_1}{A_2}}$$

Relations between (A_1, A_2) & (A, ϕ)



Pythagorean theorem

hypotenuse