② Solution and vibratory property

Model in general form:

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = 0 \quad ①$$

$\boxed{a\text{'s} > 0}$

ICs

$y(0) = y_0$

$\dot{y}(0) = \dot{y}_0$

$y(t) = ?$

Rewrite Eq. ① in another form by dividing a_2 through:

$$\ddot{y} + \frac{a_1}{a_2} \dot{y} + \frac{a_0}{a_2} y = 0 \quad ②$$

Recall

$$\omega_n^2 = \frac{a_0}{a_2} \quad (3)$$

Rewrite a_1/a_2 as

$$\frac{a_1}{a_2} = 2\omega_n \left(\frac{1}{2\omega_n} \frac{a_1}{a_2} \right)$$

define

$$\boxed{\frac{1}{2\omega_n} \frac{a_1}{a_2} \equiv \xi} = \boxed{\text{damping ratio}} \quad (4)$$

Eq. (2) $\Rightarrow$

$$\boxed{\ddot{y} + 2\omega_n \xi \dot{y} + \omega_n^2 y = 0} \quad (5)$$

Two-parameter model w/ ω_n and ξ.

We work with Eq. (5)

$y(t) = ?$

Propose a solution by physical consideration:

$$y(t) = A \sin(\omega_n t + \phi) \ ?$$

No, because $y(t)$ must decrease to zero in time.

How about

$$y(t) = A e^{-at} \sin(\omega_d t + \phi) \ ? \quad a > 0$$

May not work because if damping is large there might not be any free vibration.

How about $\boxed{y(t) = A e^{st}}$? (6)

where s may be a complex number, $s = -a + ib$

If s = negative real, $y(t)$ decays in time

s = complex w/ negative real part

$i = \sqrt{-1}$

$j = \sqrt{-1}$

book

$$e^{st} = e^{(-a+ib)t} = e^{-at} e^{ibt} = e^{-at}(\cos bt + i \sin bt)$$

$\Rightarrow$ decayed vibration

Feels OK but unclear of "i". Try it any way.

Verify

a): $y(t)$ must satisfy Eq. 5

$\because \; \dot{y}(t) = \frac{d}{dt}(Ae^{st}) = Ase^{st} = sy$

$\ddot{y} = \frac{d}{dt}(\dot{y}) = \frac{d}{dt}(Ase^{st}) = As^2e^{st} = s^2y$

plug into Eq. 5 $\Rightarrow$

$$s^2y + 2\omega_n\xi sy + \omega_n^2 y \stackrel{?}{=} 0$$

or $$\boxed{(s^2 + 2\omega_n\xi s + \omega_n^2)\underbrace{y}_{\neq 0} \stackrel{?}{=} 0} \quad 7$$

So, s must satisfy Eq. 7 for Eq. 6 to qualify as a solution

$\boxed{\text{Eq. 7 is called the characteristic eq. of the system}}$

Solving Eq. 7 for s: $$s_{1,2} = \frac{-2\omega_n\xi \pm \sqrt{(2\omega_n\xi)^2 - 4\omega_n^2}}{2}$$

Or $$\boxed{s_{1,2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n} \quad 8$$

s_1 ($+$), s_2 ($-$)

Three possible expressions for $y(t)$ depending on value of ξ.

case 1 $\boxed{\xi > 1.0} \xRightarrow{\text{Eq. 8}} s_1, s_2$ real and $s_1 \neq s_2$

Then Eq. 6 $\Rightarrow$

$$y_1(t) = A_1 e^{s_1 t}$$

$$\text{or } \; y_2(t) = A_2 e^{s_2 t}$$

either satisfies Eq. 5

But, God says $y(t)$ must be unique.

How about $y(t) = y_1(t) + y_2(t)$?

verify to satisfy Eq. 5

$$\Rightarrow (\ddot{y}_1 + \ddot{y}_2) + 2\xi\omega_n(\dot{y}_1 + \dot{y}_2) + \omega_n^2(y_1 + y_2) \stackrel{?}{=} 0$$

$$\Rightarrow (\ddot{y}_1 + 2\xi\omega_n\dot{y}_1 + \omega_n^2 y_1) + (\ddot{y}_2 + 2\xi\omega_n\dot{y}_2 + \omega_n^2 y_2) \stackrel{?}{=} 0 \checkmark$$

$$\therefore \boxed{y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \quad (9)$$

$s_{1,2}$ defined in Eq. 8 w/ $\xi > 1.0$ and $s_{1,2} < 0$

b) $y(t)$ must also satisfy ICs

Eq. (9) $\Rightarrow$

$$\left.\begin{aligned} y(0) &= A_1 + A_2 = y_0 \\ \dot{y}(0) &= A_1 s_1 + A_2 s_2 = \dot{y}_0 \end{aligned}\right\} \Rightarrow \text{determine } A_1 \text{ and } A_2$$

~~Solving $\Rightarrow$~~

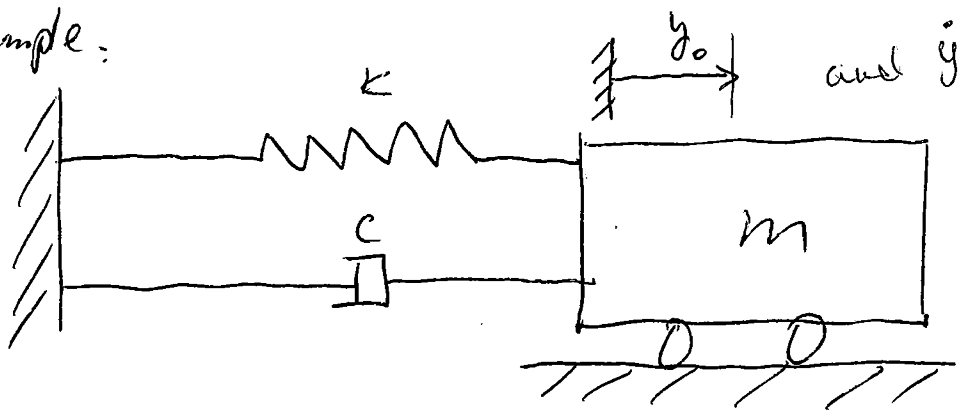
Yes, Eq. (9) is the right expression for $\zeta_1 > 1.0$

Solving A_1 and A_2 $\Rightarrow$

$$A_1 = \frac{y_0 \omega_n (\xi + \sqrt{\xi^2 - 1}) + \dot{y}_0}{2 \omega_n \sqrt{\xi^2 - 1}} \quad (10)$$

$$A_2 = \frac{-y_0 \omega_n (\xi - \sqrt{\xi^2 - 1}) - \dot{y}_0}{2 \omega_n \sqrt{\xi^2 - 1}} \quad (11)$$

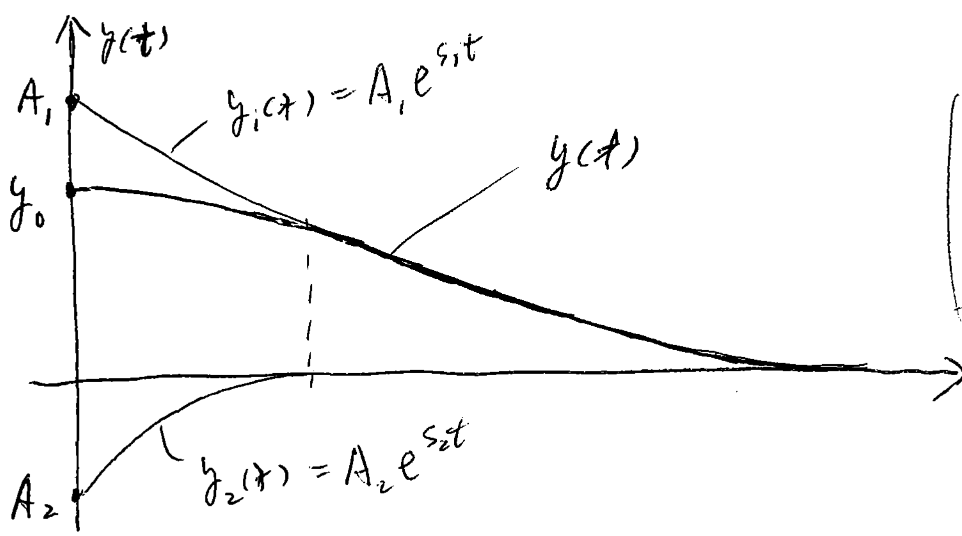
Example:



and $\dot{y}_0 = 0$

$y(t) = ?$

$$\xi = \frac{1}{2\sqrt{\frac{k}{m}}} \frac{c}{m} > 1.0$$



$$s_1 = (-\xi + \sqrt{\xi^2 - 1})\,\omega_n$$

$$s_2 = (-\xi - \sqrt{\xi^2 - 1})\,\omega_n$$

No vibration under its own.

$$\boxed{\xi > 1.0 \Rightarrow \text{overdamped system}}$$

case ② $\boxed{\xi = 1.0}$ $\overset{\text{Eq. ⑧}}{\Longrightarrow}$ $s_1 = s_2 = -\omega_n$

Then Eq. ⑥ $\Rightarrow$ $y_1(t) = A_1 e^{-\omega_n t}$ —— (can't satisfy ICs alone)

and $y_2(t) = A_2 t e^{-\omega_n t}$ —— (can be verified to satisfy Eq. ⑤)

So,

$$y(t) = A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t}$$

b) Honor ICs $\Rightarrow$

$$y(0) = A_1 = y_0$$

$$\dot{y}(0) = -A_1 \omega_n + A_2 = \dot{y}_0$$

$\Rightarrow$

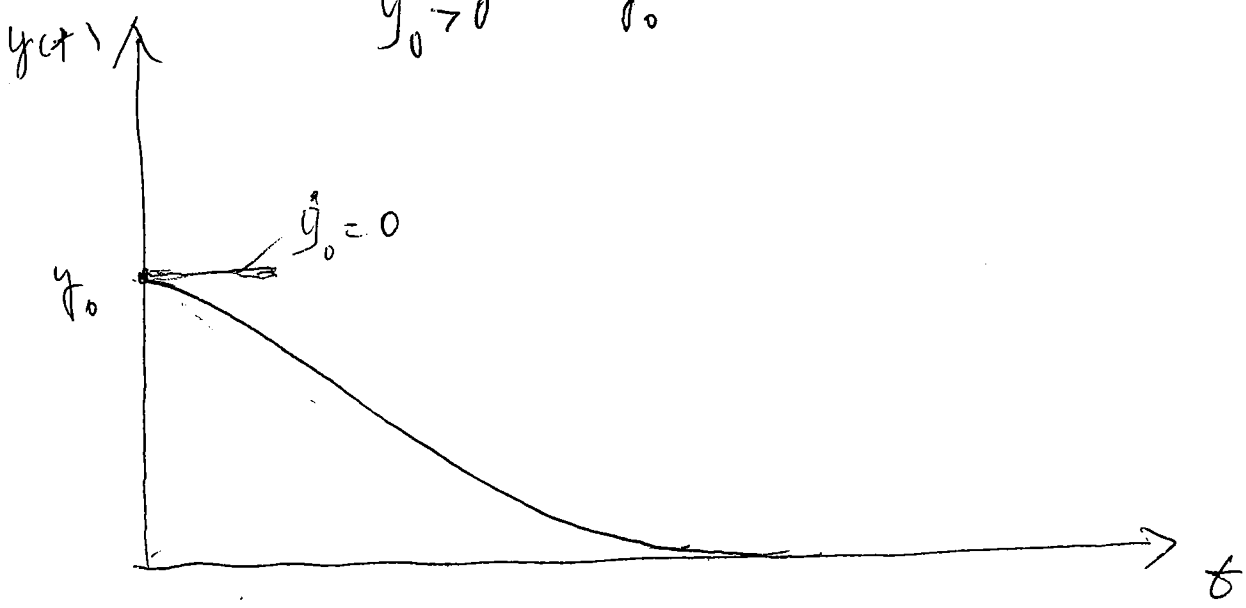
$$\boxed{\begin{aligned} A_1 &= y_0 \\ A_2 &= \dot{y}_0 + y_0 \omega_n \end{aligned}} \qquad (12)$$

$\therefore$

$$\boxed{y(t) = [y_0 + (\dot{y}_0 + \omega_n y_0) t] e^{-\omega_n t}} \qquad (13)$$

Response to the example,

$y_0 > 0 \quad \dot{y}_0 = 0$



No vibration under its own.

$\xi = 1.0 \Rightarrow$ critically damped system

Case 3 $\boxed{0 < \xi < 1.0}$ Eq. 8 $\Rightarrow$ $s_{1,2}$ are complex conjugates

$$s_{1,2} = -\omega_n \xi \pm i\omega_n\sqrt{1-\xi^2}, \quad i = \sqrt{-1}$$

Eq. 6 $\Rightarrow$ $y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$$= A_1 e^{(-\omega_n\xi + i\omega_n\sqrt{1-\xi^2})t} + A_2 e^{(-\omega_n\xi - i\overbrace{\omega_n\sqrt{1-\xi^2}}^{\omega_d})t}$$

$$= e^{-\omega_n\xi t}\left(A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right)$$

$e^{i\omega_d t} = \cos\omega_d t + i\sin\omega_d t$

$e^{-i\omega_d t} = \cos(-\omega_d t) + j\sin(-\omega_d t)$

$= \cos\omega_d t + j\sin\omega_d t$

$$= e^{-\xi\omega_n t}\left[\underbrace{(A_1 + A_2)}_{B_1}\cos\omega_d t + \underbrace{i(A_1 - A_2)}_{B_2}\sin\omega_d t\right]$$

$$\therefore \boxed{y(t) = e^{-\omega_n\xi t}\left(B_1\cos\omega_d t + B_2\sin\omega_d t\right)} \quad (14)$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\xi^2} = \text{damped natural frequency}} \quad (15)$$

b) $y(t)$ must honor ICs

Eq. 14 $\Rightarrow$

$$\left.\begin{aligned} y(0) &= B_1 = y_0 \\ \dot{y}(0) &= -\omega_n\xi B_1 + B_2\omega_d = \dot{y}_0 \end{aligned}\right\} \Rightarrow B_1 \,\&\, B_2$$

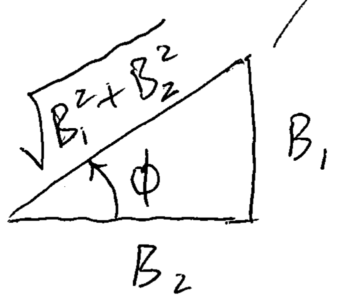
$$\Rightarrow \quad \boxed{\begin{aligned} B_1 &= y_0 \\ B_2 &= \frac{\dot{y}_0 + \omega_n \xi y_0}{\omega_d} \end{aligned}} \qquad (16)$$

An alternative expression for $y(t)$

Eq. (14) $\Rightarrow$

$$y(t) = \sqrt{B_1^2 + B_2^2}\, e^{-\omega_n \xi t} \left[\underbrace{\frac{B_1}{\sqrt{B_1^2 + B_2^2}}}_{\sin\phi} \cos\omega_d t + \underbrace{\frac{B_2}{\sqrt{B_1^2 + B_2^2}}}_{\cos\phi} \sin\omega_d t \right]$$

Define a reference triangle.



Let $A = \sqrt{B_1^2 + B_2^2}$

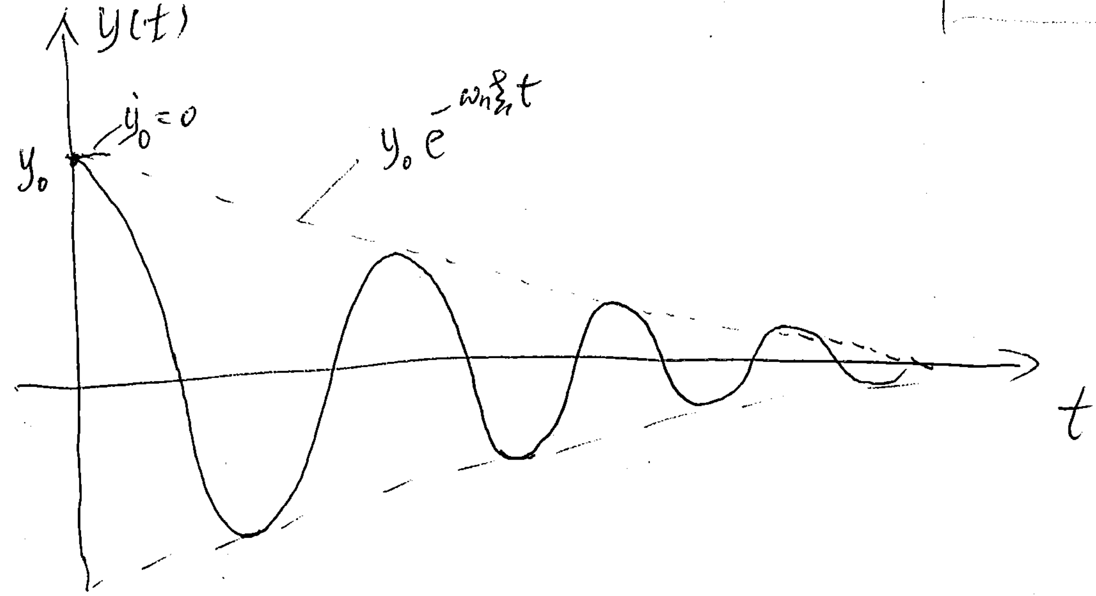
$$\therefore \quad \boxed{y(t) = A e^{-\omega_n \xi t} \sin(\omega_d t + \phi)} \qquad (14')$$

$$\boxed{\begin{aligned} A &= \sqrt{B_1^2 + B_2^2} \\ \phi &= \tan_2^{-1} \frac{B_1}{B_2} \end{aligned}} \qquad (16.1)$$

The example again: $y_0 > 0$, $\dot{y}_0 = 0$ $\Rightarrow \phi = \tan_2^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$

$y(0) = A \sin\phi \Rightarrow A = \frac{y_0}{\sin\phi}$

No need



$0 < \xi < 1.0$ develops free vibration

$\Rightarrow$ underdamped system

If $\xi = 0$, $\omega_d = \omega_n$, Eq. (14) $\Rightarrow$

$$y(t) = A \sin(\omega_n t + \phi)$$

$\Rightarrow$ undamped system.