② Solution of system response

Math model in standard form:

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = F_0 \sin\omega t \quad ①$$

or $\cos\omega t$

or $\sin(\omega t + \beta)$

⋮

ICs:

$$y(0) = y_0$$

$$\dot{y}(0) = \dot{y}_0$$

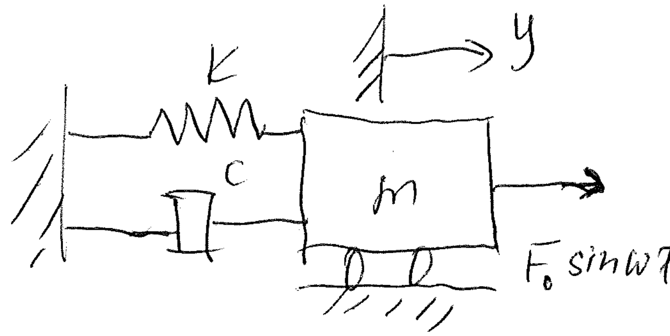
As in section 2.2,

$$\frac{\text{Eq. ①}}{a_2} \Rightarrow \boxed{\ddot{y} + 2\omega_n \zeta \dot{y} + \omega_n^2 y = E \sin\omega t} \quad ②$$

$$\boxed{E = \frac{F_0}{a_2} = \text{magnitude of harmonic excitation.}}$$

$y(t) = ?$ — must satisfy Eq. ② and ICs.

Trial solution



$$y(t) = Y \sin(\omega t + \psi) \quad (3)$$

a) Satisfying Eq. (2) ?

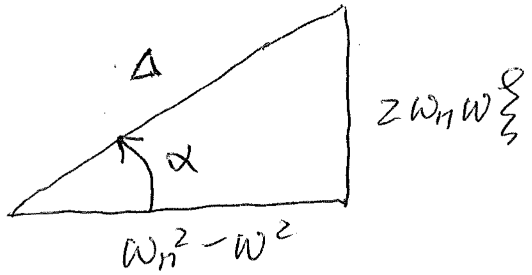
Eq (2) $\Rightarrow$

$$-Y\omega^2 \sin(\omega t + \psi) + 2\omega_n \xi Y \omega \cos(\omega t + \psi) + \omega_n^2 Y \sin(\omega t + \psi) \overset{?}{=} E \sin \omega t$$

Or

$$Y\left[(\omega_n^2 - \omega^2)\sin(\omega t + \psi) + 2\omega_n \xi \omega \cos(\omega t + \psi)\right] \overset{?}{=} E \sin \omega t \quad (4)$$

Define a ref. triangle:



$$\Delta = \left[(\omega_n^2 - \omega^2)^2 + (2\xi \omega_n \omega)^2\right]^{1/2}$$

$$\alpha = \tan_2^{-1} \frac{2\xi \omega_n \omega}{\omega_n^2 - \omega^2}$$

Eq. (4) $\Rightarrow$

$$Y\Delta\left[\frac{\omega_n^2-\omega^2}{\Delta}\sin(\omega t+\psi)+\frac{2\zeta\omega_n\omega}{\Delta}\cos(\omega t+\psi)\right]$$

$$\stackrel{?}{=} E\sin\omega t$$

Or

$$(Y\Delta)\sin(\omega t+\psi+\alpha)\stackrel{?}{=}E\sin\omega t$$

Yes if and only if:

$$Y\Delta = E$$

and $\omega t+\psi+\alpha=\omega t$ or $\psi=-\alpha$

$$\Rightarrow \boxed{Y=\frac{E}{[(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2]^{1/2}}} \quad (5)$$

and

$$\boxed{\psi=-\tan_2^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}} \quad (6)$$

So, Y and ψ in the trial solution (Eq. (3)) must satisfy Eqs. (5) and (6)

Define :

$$\boxed{r = \frac{\omega}{\omega_n}} \qquad (7)$$

Eq. (5) $\Rightarrow$

$$\boxed{Y = \frac{E/\omega_n^2}{\left[(1-r^2)^2 + (2\zeta r)^2\right]^{1/2}}} \qquad (M)$$

Eq. (6) $\Rightarrow$

$$\boxed{\psi = -\tan_2^{-1} \frac{2\zeta r}{1-r^2}} \qquad (P)$$

If the excitation on the RHS of Eq. (2) is.

$$E \sin(\omega t + \beta) \xrightarrow{\text{then}} y(t) = Y \sin(\omega t + \beta + \psi) \qquad (8)$$

$$E \cos(\omega t) \longrightarrow y(t) = Y \cos(\omega t + \psi) \qquad (8)$$

$\vdots$

b) Satisfying ICs ?

Too bad, the trial solution Eq. (3) can't satisfy ICs, simply because $y(0) = Y \sin\phi$ with Y and ϕ determined by Eqs (M) and (P) can't be made equal to y_0 in general.

A new trial solution:

$$\boxed{y(t) = y_h(t) + y_p(t)} \quad (8)$$

where $y_p(t)$ — expression by Eq. (3)

$y_h(t)$ — expression from Sec. 2.2 in notes.

$$y_h(t) = \begin{cases} A_1 e^{s_1 t} + A_2 e^{s_2 t} & \xi > 1.0 \\ (A_1 + A_2 t) e^{-\omega_n t} & \xi = 1.0 \\ e^{-\xi \omega_n t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) & 0 < \xi < 1.0 \end{cases}$$

$y(t)$ satisfying Eq. ② ?

Eq. ⑧ into Eq. ② $\Rightarrow$

$$(\ddot{y}_h + \ddot{y}_p) + 2\xi\omega_n(\dot{y}_h + \dot{y}_p) + \omega_n^2(y_h + y_p) \overset{?}{=} E\sin\omega t$$

or

$$\underbrace{\left(\ddot{y}_h + 2\xi\omega_n\dot{y}_h + \omega_n^2 y_h\right)}_{=0 \text{ by Section 2.2}} + \underbrace{\left(\ddot{y}_p + 2\xi\omega_n\dot{y}_p + \omega_n^2 y_p\right)}_{\text{satisfied by Eq. ③}} \overset{?}{=} E\sin\omega t$$

✓

$y(t)$ satisfying ICs ?

Eq. ⑧ $\Rightarrow$

$$\begin{aligned} y(0) &= \boxed{y_h(0) + y_p(0) = y_0} \\ \dot{y}(0) &= \boxed{\dot{y}_h(0) + \dot{y}_p(0) = \dot{y}_0} \end{aligned} \quad ⑨ \Rightarrow \text{determine } A_1, A_2 \text{ (or } B_1, B_2) \text{ in } y_h(t)$$

Yes, $y(t)$ is the guy.

Eq. ⑨ $\Rightarrow$

$$\boxed{\begin{aligned} y_h(0) &= y_0 - y_p(0) = \cancel{y_0^*} = y_{h0} \\ \dot{y}_h(0) &= \dot{y}_0 - \dot{y}_p(0) = \cancel{\dot{y}_0^*} = \dot{y}_{h0} \end{aligned}}$$

contain two unknowns

known

$y_p(0)$, $\dot{y}_p(0)$ from Eq. ③ (w/ excitation $E\sin\omega t$)

So, we may determine the two constants in $y_h(t)$ using equations obtained in Section 2.2 by replacing $\boxed{y_0 \text{ by } \underbrace{y_0 - y_p(0)}_{y_0^*,\ y_{h0}} \text{ and } \dot{y}_0 \text{ by } \underbrace{\dot{y}_0 - \dot{y}_p(0)}_{\dot{y}_0^*,\ \dot{y}_{h0}}}$

For $\zeta > 1.0$,

A_1, A_2 by Eqs. 10 and 11 in Section 2.2 w/

For $\zeta = 1.0$

A_1, A_2 by Eq. 12:

$$\boxed{\begin{aligned} A_1 &= (y_0 - y_p(0)) = y_0^* \\ A_2 &= (\dot{y}_0 - \dot{y}_p(0)) + (y_0 - y_p(0))\,\omega_n \end{aligned}}$$

For $0 \le \zeta < 1.0$

B_1 and B_2 by Eq. 16:

$$\boxed{\begin{aligned} B_1 &= y_0 - y_p(0) \\ B_2 &= \frac{(\dot{y}_0 - \dot{y}_p(0)) + \omega_n \zeta (y_0 - y_p(0))}{\omega_d} \end{aligned}}$$

So, $\boxed{y(t) = y_h(t) + y_p(t)}$ (8)

$y_h(t)$ = natural response

$y_p(t) = Y \sin(\omega t + \psi)$ = follow-up response

($E \sin \omega t$)

$y(t)$ = total response.

As time goes large, $y_h(t) \to 0$

$\Rightarrow \; y(t) = y_p(t)$

We are often interested in $y_p(t)$ for a given driving action: $E \sin \omega t$, $E \cos \omega t$, or $E \sin(\omega t + \beta)$

Or $\boxed{Y, \psi \sim E, \omega \text{ relations, by Eqs. (m) and (P)}}$