Homework 11

Due: *In class*, Friday Nov. 16

1. [46 pts] The math model of the system in Prob. 2 of hw10 is given by

$$\begin{bmatrix} m_1+m_2 & m_2L \\ m_2L & J_2+L^2m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix}+\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{\theta} \end{bmatrix}+\begin{bmatrix} k & 0 \\ 0 & m_2gL \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $L = 0.4\,\text{m}$, $k = 200\,\text{N/m}$, $J_2 = 0$, $c = 0$, and $m_1 = m_2 = 2.0\,\text{kg}$.

(1) (2 pts) The 1st natural frequency of the system in rad/s is

a) $\omega_{n1} = 1.37$ b) $\omega_{n1} = 2.56$

c) $\omega_{n1} = 4.34$ d) $\omega_{n1} = 7.83$

(2) (2 pts) The 2nd natural frequency of the system in rad/s is

a) $\omega_{n1} = 6.14$ b) $\omega_{n1} = 8.21$

c) $\omega_{n1} = 10.11$ d) $\omega_{n1} = 11.41$

(3) (4 pts) The 1st mode shape of the system is

a) $\begin{bmatrix} 1 \\ 0.721 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ -0.997 \end{bmatrix}$

c) $\begin{bmatrix} 0.120 \\ 0.993 \end{bmatrix}$ d) $\begin{bmatrix} 0.409 \\ -0.913 \end{bmatrix}$

(4) (4 pts) The 2nd mode shape of the system is

a) $\begin{bmatrix} 1 \\ -0.451 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ 0.607 \end{bmatrix}$

c) $\begin{bmatrix} 0.212 \\ 0.977 \end{bmatrix}$ d) $\begin{bmatrix} 0.309 \\ -0.951 \end{bmatrix}$

(5) (3 pts) Write down the general expression for the free vibration of the undamped system with four undetermined constants: $X^{(1)}$, ϕ_1, $X^{(2)}$ and ϕ_2.

$$\underline{X}(t) = \begin{bmatrix} 0.120 \\ 0.993 \end{bmatrix} X^{(1)} \sin(4.34t + \phi_1) + \begin{bmatrix} 0.309 \\ -0.951 \end{bmatrix} X^{(2)} \sin(11.41t + \phi_2)$$

(6) (3 pts) Suppose you lift the pendulum with an angle of $\theta(0) = 2.0$ rad. You then release it with $\dot{\theta}(0) = -1.0$ rad/s to initiate a free vibration of the system. Write down the initial conditions for the problem.

$\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 2.0 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} \dot{x}_1 \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -1.0 \end{bmatrix}$

(7) (4 pts) The two constants for the 1st mode in the free-vibration expression yield the following:

a) $X^{(1)} sin\phi_1$= 0.845 $\qquad$ b) $X^{(1)} sin\phi_1$=-0.903

c) $X^{(1)} sin\phi_1$= 1.469 $\qquad$ d) $X^{(1)} sin\phi_1$=-1.787

(8) (4 pts) The two constants for the 2nd mode in the free-vibration expression yield the following:

a) $X^{(2)} sin\phi_2$= -1.508 $\qquad$ b) $X^{(2)} sin\phi_2$=2.124

c) $X^{(2)} sin\phi_2$=-0.570 $\qquad$ d) $X^{(2)} sin\phi_2$=1.213

(9) (4 pts) The magnitudes of the two modes in the free vibration expression of Part (5) are:

a) $X^{(1)}$= 1.478 $\quad$ $X^{(2)}$=-0.443 $\qquad$ b) $X^{(1)}$=-1.128 $\quad$ $X^{(2)}$=-0.443

c) $X^{(1)}$= 1.128 $\quad$ $X^{(2)}$=0.570 $\qquad$ d) $X^{(1)}$= 1.478 $\quad$ $X^{(2)}$=0.570

(10) (4 pts) The phase angles of the two modes in the free vibration expression of Part (5) are:

a) ϕ_1=1.232 $\quad$ ϕ_2=1.527 $\qquad$ b) ϕ_1=-1.232 $\quad$ ϕ_2=-1.151

c) ϕ_1=1.685 $\quad$ ϕ_2=-1.151 $\qquad$ d) ϕ_1=1.685 $\quad$ ϕ_2=-1.527

(11) (12 pts) Program in matlab the expressions of $x(t)$ and $\theta(t)$ you obtain to plot the free vibration of the system for a time period of $t = 3\pi/\omega_{n1}$. Use the **subplot** feature in the matlab to plot $x\ vs.t$ on top and $\theta\ vs.t$ on bottom of the graph with labels. Use sufficient data points to generate smooth $x(t)$ and $\theta(t)$ curves with plotting grids, *making sure ICs are satisfied to be correct.* Examine the plot to envision how the cart moves and pendulum swings spontaneously with two modes of contributions. Submit your program along with the plot.

(5) (3 pts) Write down the general expression for the free vibration of the undamped system with four undetermined constants: $X^{(1)}$, ϕ_1, $X^{(2)}$ and ϕ_2.

$$\mathbf{x(t)} = \begin{bmatrix} 0.120 \\ 0.993 \end{bmatrix} X^{(1)} \sin(4.34t + \Phi_1) + \begin{bmatrix} 0.309 \\ -0.951 \end{bmatrix} X^{(2)} \sin(11.41t + \Phi_2)$$

(6) (3 pts) Suppose you lift the pendulum with an angle of $\theta(0) = 2.0$ rad. You then release it with $\dot{\theta}(0) = -1.0$ rad/s to initiate a free vibration of the system. Write down the initial conditions for the problem.

$$\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(7) (4 pts) The two constants for the 1st mode in the free-vibration expression yield the following:

a) $X^{(1)} sin\phi_1$= 0.845 b) $X^{(1)} sin\phi_1$=-0.903

c) $X^{(1)} sin\phi_1$= 1.469 d) $X^{(1)} sin\phi_1$=-1.787

(8) (4 pts) The two constants for the 2nd mode in the free-vibration expression yield the following:

a) $X^{(2)} sin\phi_2$= -1.508 b) $X^{(2)} sin\phi_2$=2.124

c) $X^{(2)} sin\phi_2$=-0.570 d) $X^{(2)} sin\phi_2$=1.213

(9) (4 pts) The magnitudes of the two modes in the free vibration expression of Part (5) are:

a) $X^{(1)}$= -1.775 $X^{(2)}$=-0.762 b) $X^{(1)}$=-1.521 $X^{(2)}$=-0.443

c) $X^{(1)}$= 1.128 $X^{(2)}$=0.324 d) $X^{(1)}$= 1.478 $X^{(2)}$=0.570

(10) (4 pts) The phase angles of the two modes in the free vibration expression of Part (5) are:

a) ϕ_1=1.772 ϕ_2=1.045 b) ϕ_1=-1.232 ϕ_2=-1.432

c) ϕ_1=1.324 ϕ_2=-1.151 d) ϕ_1=1.685 ϕ_2=-1.527

(11) (12 pts) Program in matlab the expressions of $x(t)$ and $\theta(t)$ you obtain to plot the free vibration of the system for a time period of $t = 3\pi/\omega_{n1}$. Use the **subplot** feature in the matlab to plot $x\ vs. t$ on top and $\theta\ vs. t$ on bottom of the graph with labels. Use sufficient data points to generate smooth $x(t)$ and $\theta(t)$ curves with plotting grids, *making sure ICs are satisfied to be correct*. Examine the plot to envision how the cart moves and pendulum swings spontaneously with two modes of contributions. Submit your program along with the plot.

1. sol: ① "hand calculation"

(1)(2): Substituting the values into system math model, we get:

$$\begin{bmatrix} 4 & 0.8 \\ 0.8 & 0.32 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 200 & 0 \\ 0 & 7.84 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \cdots ①$$

i.e. $[M]\ddot{\bar{x}} + [K]\bar{x} = \bar{0}$

For free vibration: $\bar{X}(t) = \bar{X}\sin(\omega_n t + \phi)$ --- ②

Equation ② into ① ⇒

$$[[K] - \omega_n^2 [M]]\bar{X} = \bar{0} \quad \cdots ③$$

System characteristic equation is:

$$\text{Det}[[K] - \omega_n^2[M]] = 0$$

i.e. $\text{Det}\begin{bmatrix} 200-4\omega_n^2 & -0.8\omega_n^2 \\ -0.8\omega_n^2 & 7.84-0.32\omega_n^2 \end{bmatrix} = (200-4\omega_n^2)(7.84-0.32\omega_n^2) - (0.8\omega_n^2)^2 = 0$

⇒ $\omega_{n1}^2 = 18.82$, $\omega_{n2}^2 = 130.18$

⇒ $\boxed{\omega_{n1} = 4.34, \quad \omega_{n2} = 11.41}$

(3)(4): put ω_{n1} into ③, we get

$$\begin{bmatrix} 124.72 & -15.06 \\ -15.06 & 1.82 \end{bmatrix}\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let $\bar{X}_1^{(1)} = 1$, ⇒ $\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(1)} = \begin{bmatrix} 1 \\ 8.28 \end{bmatrix}$ $\xrightarrow{\text{normalize}}$ $\boxed{\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(1)} = \begin{bmatrix} 0.120 \\ 0.993 \end{bmatrix}}$ 1st mode shape

put ω_{n2} into ③, we get

$$\begin{bmatrix} -320.75 & -104.14 \\ -104.14 & -33.82 \end{bmatrix}\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let $\bar{X}_1^{(2)} = 1$, ⇒ $\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(2)} = \begin{bmatrix} 1 \\ -3.08 \end{bmatrix}$ $\xrightarrow{\text{normalize}}$ $\boxed{\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(2)} = \begin{bmatrix} 0.309 \\ -0.951 \end{bmatrix}}$ 2nd mode shape.

(5): Thus, the vibration expression is:

$$\boxed{\bar{X}(t) = \begin{bmatrix} x_1(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} 0.120 \\ 0.993 \end{bmatrix}\bar{X}^{(1)}\sin(4.34t + \phi_1) + \begin{bmatrix} 0.309 \\ -0.951 \end{bmatrix}\bar{X}^{(2)}\sin(11.41t + \phi_2)} \quad \cdots ④$$

(6): since $\theta(0)=2$ rad, $\dot{\theta}(0)=-1.0$ rad/s, and $x(0)=0$, $\dot{x}(0)=0$

$\Rightarrow$ initial conditions $\begin{Bmatrix} x_1 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2.0 \end{Bmatrix}$, $\begin{Bmatrix} \dot{x}_1 \\ \dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1.0 \end{Bmatrix}$

(7)(8): According to ④

$$\bar{X}(0) = \begin{bmatrix} 0.120 \\ 0.993 \end{bmatrix} X^{(1)}\sin\phi_1 + \begin{bmatrix} 0.309 \\ -0.951 \end{bmatrix} X^{(2)}\sin\phi_2 = \begin{bmatrix} 0.120 & 0.309 \\ 0.993 & -0.951 \end{bmatrix} \begin{bmatrix} X^{(1)}\sin\phi_1 \\ X^{(2)}\sin\phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.0 \end{bmatrix}$$

$$\Rightarrow \boxed{X^{(1)}\sin\phi_1 = 1.469, \quad X^{(2)}\sin\phi_2 = -0.570} \quad \cdots ⑤$$

(9)(10): And

$$\dot{\bar{X}}(0) = \begin{bmatrix} 0.120 \\ 0.993 \end{bmatrix} \omega_{n1} X^{(1)}\cos\phi_1 + \begin{bmatrix} 0.309 \\ -0.951 \end{bmatrix} \omega_{n2} X^{(2)}\cos\phi_2 = \begin{bmatrix} 0.120 & 0.309 \\ 0.993 & -0.951 \end{bmatrix} \begin{bmatrix} \omega_{n1} X^{(1)}\cos\phi_1 \\ \omega_{n2} X^{(2)}\cos\phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.0 \end{bmatrix}$$

$$\Rightarrow \omega_{n1} X^{(1)}\cos\phi_1 = -0.734, \quad \omega_{n2} X^{(2)}\cos\phi_2 = 0.285$$

$$\Rightarrow \boxed{X^{(1)}\cos\phi_1 = \frac{-0.734}{4.34} = -0.170, \quad X^{(2)}\cos\phi_2 = 0.025} \quad \cdots ⑥$$

From ⑤ and ⑥, we get

$$\boxed{\phi_1 = \tan_2^{-1}\frac{X^{(1)}\sin\phi_1}{X^{(1)}\cos\phi_1} = \tan_2^{-1}\frac{1.469}{-0.170} = 1.685} \quad \cdots ⑦$$

$$\text{And } \boxed{X^{(1)} = \frac{1.469}{\sin\phi_1} = \frac{1.469}{\sin 1.685} = 1.478} \quad \cdots ⑧$$

$$\boxed{\phi_2 = \tan_2^{-1}\frac{X^{(2)}\sin\phi_2}{X^{(2)}\cos\phi_2} = \tan_2^{-1}\frac{-0.570}{0.025} = -1.527} \quad \cdots ⑨$$

$$\boxed{X^{(2)} = \frac{-0.570}{\sin\phi_2} = \frac{-0.570}{\sin(-1.527)} = 0.570} \quad \cdots ⑩$$

(11). substitute ⑦⑧⑨⑩ into ④, get vibration expression.

$$\bar{x}(t) = \begin{Bmatrix} x_1(t) \\ \theta(t) \end{Bmatrix} = \begin{bmatrix} 0.120 \\ 0.993 \end{bmatrix} 1.478\sin(4.34t + 1.685) + \begin{bmatrix} 0.309 \\ -0.951 \end{bmatrix} 0.570\sin(11.41t - 1.527)$$

The matlab code and graph are shown on the next page.

Problem 1.11

```
%{
 * Course:ME370
 * Name:Liming Gao
 * Date: Nov. 07, 2018
 *
 * Program Description: Problem 1.11, hw11.
 %}

clear all
close all

wn1 = 4.34; % 1st natural frequency
wn2 = 11.41; %2nd natural frequency
X = [1.478 0.570]; % magnitude ratio
P = [1.685 -1.527]; % initial phase
S = [0.120 0.309 ; 0.993 -0.951]; % mode shape

t= 0:0.001:3*pi/wn1; %Set time of two periods
x = S(1,1)*X(1)*sin(wn1*t+P(1)) + S(1,2)*X(2)*sin(wn2*t+P(2));
theta = S(2,1)*X(1)*sin(wn1*t+P(1)) + S(2,2)*X(2)*sin(wn2*t+P(2));

figure(1)
subplot(2,1,1)
plot(t,x,'b','LineWidth',1); %plot the curve

xlabel('Time(sec)')
ylabel('x (m)')
grid on

subplot(2,1,2)
plot(t,theta,'r','LineWidth',1); %plot the curve

xlabel('Time(sec)')
ylabel('\theta (rad)')
grid on
```

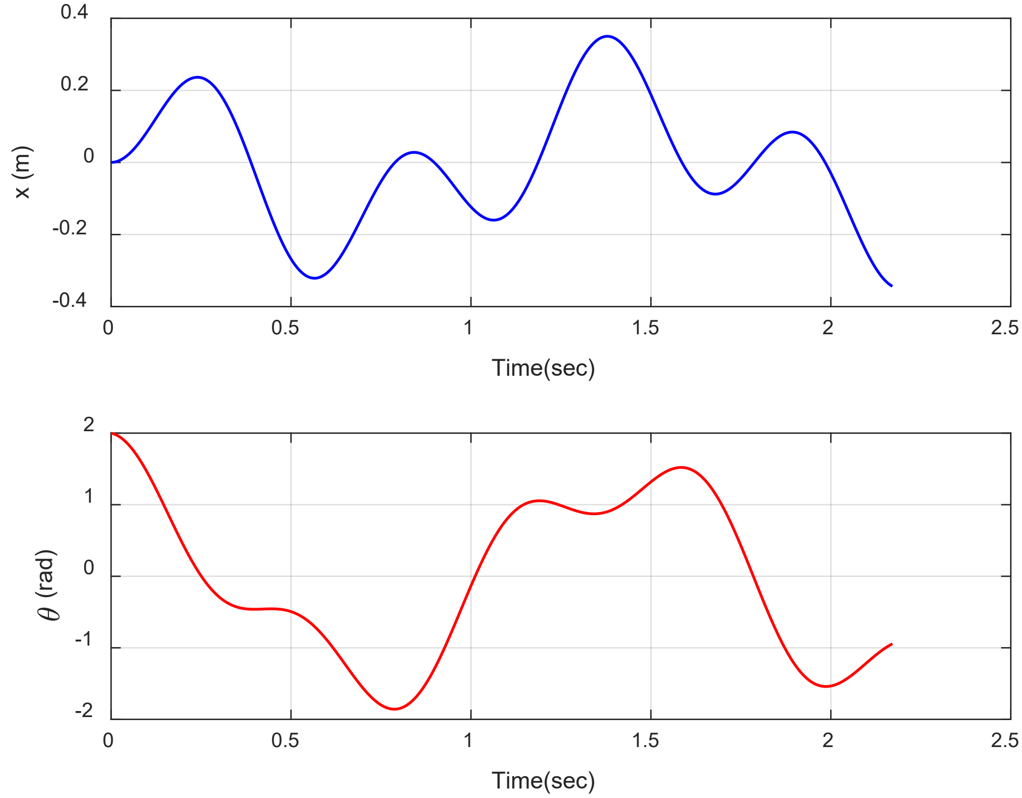

Figure 1 The vibration of the system

2. [10 pts] Consider a system with the following governing equations:

$$\begin{bmatrix} 10 & 5 & 0 \\ 5 & 50 & 10 \\ 0 & 10 & 25 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{bmatrix} + \begin{bmatrix} 1000 & -120 & -270 \\ -120 & 1500 & 90 \\ -270 & 90 & 400 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} + \begin{bmatrix} 2\times 10^5 & -10^5 & 0 \\ -10^5 & 2\times 10^5 & 0 \\ 0 & 0 & 5\times 10^5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use Matlab to determine the natural frequencies and the mode shapes of the system. Include your Matlab program along with the results.

Code (ignore the damping)

```
clear all
close all
M = [10 5 0; 5 50 10; 0 10 25]; % mass matrices
C=[1000 -120 -270; -120 1500 90; -270 90 400]; % damping matrices
K = 10^5*[2 -1 0 ; -1 2 0; 0 0 5]; % stiffness matrices

A = inv(M)*K; %system matrix
[v,d]=eig(A); %obtain eigenvalues and eigenvectors of A

%sort eigenvalues and eigenvectors
[d_sorted, index] = sort(diag(d), 'ascend');
v_sorted = v(:,index);

%obtain natural frequencies and mode shapes
w1= sqrt(d_sorted(1))
w2= sqrt(d_sorted(2))
w3= sqrt(d_sorted(3))
ms1=v_sorted(:,1)
ms2=v_sorted(:,2)
ms3=v_sorted(:,3)
```

Results:

w1 = 50.202

w2 =141.42

w3 =165.42

ms1 =	ms2 =	ms3 =
-0.54091	-0.70711	-0.87359
-0.83969	3.3323e-16	0.27167
-0.048426	0.70711	-0.40377

Code (not ignore the damping)

```
clear all
close all
M = [10 5 0; 5 50 10; 0 10 25]; % mass matrices
C=[1000 -120 -270; -120 1500 90; -270 90 400]; % damping matrices
K = 10^5*[2 -1 0 ; -1 2 0; 0 0 5]; % stiffness matrices

MI = inv(M); % inverse matices of mass matrices

R=[zeros(3,3) eye(3);-MI*K -MI*C]; %state matrices

[V, D]=eig(R); %obtain eigenvalues and eigenvectors of R

Wn= [abs(D(5,5));abs(D(3,3));abs(D(1,1))] %natural frequency
```

Result:

```
Wn =

        50.428

        151.59
        153.63
```