Example: Two DOF damped system

Model:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} e^{i\omega t} \qquad (1)$$

Follow-up response:

$$x(t) = \begin{bmatrix} \bar{X}_1^* \\ \bar{X}_2^* \end{bmatrix} e^{i\omega t} \qquad (2)$$

Eq. (2) into Eq. (1) $\Rightarrow$

$$\left[\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}(-\omega^2) + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}(i\omega) + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\right]\begin{bmatrix} \bar{X}_1^* \\ \bar{X}_2^* \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

Or
$$\begin{bmatrix} (-\omega^2 m_{11}+k_{11})+i\omega c_{11} & (-\omega^2 m_{12}+k_{12})+i\omega c_{12} \\ (-\omega^2 m_{21}+k_{21})+i\omega c_{21} & \text{" } (22) \end{bmatrix} \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

Impedance matrix

Or
$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \qquad (3)$$

$$\therefore \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

$$= \frac{1}{\det[Z]} \begin{bmatrix} Z_{22} & -Z_{12} \\ -Z_{21} & Z_{11} \end{bmatrix} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

So,
$$X_1^* = \frac{Z_{22}F_{10} - Z_{12}F_{20}}{\det[Z]} = \frac{N_1}{D} \qquad (4) \quad = X_1 e^{i\phi_1}$$

$$X_2^* = \frac{-Z_{12}F_{10} + Z_{11}F_{20}}{\mathrm{Det}[Z]} = \frac{N_2}{D} \qquad (5) \quad = X_2 e^{i\phi_2}$$

Then

$$\bar{X}_1 = \frac{Amp(N_1)}{Amp(D)} = \frac{[Re^2(N_1) + Im^2(N_1)]^{1/2}}{[Re^2(D) + Im^2(D)]^{1/2}}$$

$$\psi_1 = angle(N_1) - angle(D)$$

$$= \tan_2^{-1} \frac{Im(N_1)}{Re(N_1)} - \tan_2^{-1} \frac{Im(D)}{Re(D)}$$

and

$$\bar{X}_2 = \frac{Amp(N_2)}{Amp(D)}$$

$$\psi_2 = \tan_2^{-1} \frac{Im(N_2)}{Re(N_2)} - \tan_2^{-1} \frac{Im(D)}{Re(D)}$$

Finally:

$$\bar{x}(t) = \begin{bmatrix} \bar{X}_1 e^{i(\omega t + \psi_1)} \\ \bar{X}_2 e^{i(\omega t + \psi_2)} \end{bmatrix}$$

```matlab
%Programming in MATLAB to
%solve for a 2-DOF Harmonically-excited vibration

%Define [M],[C],[K] and right-hand side of math model
M=[11 0; 0 22]
C=[91 -10; -10 50]
K=[1000 -500; -500 2000]
F=[1;0]

%Define frequency of excitation
w=50;

%Form the impedance matrix
Z=M*(-w^2)+C*(i*w)+K

%Solve for X*
Xcmplx=inv(Z)*F

%Obtain amplitude and phase delay of vibration
X=abs(Xcmplx)
S=angle(Xcmplx)
```

Results of MATLAB run

```
M =
    11     0
     0    22

C =
    91   -10
   -10    50

K =
        1000        -500
        -500        2000

F =
     1
     0

w =
    50

Z =
  1.0e+004 *

  -2.6500 + 0.4550i  -0.0500 - 0.0500i
  -0.0500 - 0.0500i  -5.3000 + 0.2500i

Xcmplx =
  1.0e-004 *

  -0.3665 - 0.0631i
   0.0027 + 0.0042i

X =
  1.0e-004 *

    0.3719
    0.0050

S =
   -2.9712
    1.0029
```