Natural freqs and mode shapes of NDOF system

Gov. eq. (Math model)

$$[M]\ddot{\bar{x}} + [K]\bar{x} = \bar{0} \quad (1)$$

$[M]$ and $[K]$ = $n \times n$ square matrices.

Free vibration:

$$\bar{x}(t) = \bar{X}\sin(\omega_n t + \phi) \quad (2)$$

Eq. (2) into Eq. (1) $\Rightarrow$

$$-[M]\omega_n^2\bar{X}\sin(\omega_n t + \phi) + [K]\bar{X}\sin(\omega_n t + \phi) = \bar{0}$$

$$\Rightarrow \quad \left[[K] - \omega_n^2[M]\right]\bar{X} = \bar{0} \quad (3)$$

and

system characteristic eq:

$$\boxed{\text{Det}\left[[K] - \omega_n^2 [M]\right] = 0} \quad (4)$$

which determines the n natural freqs of the system

Alternatively, define:

$[A] = [M]^{-1}[K]$ = system matrix

$[M]^{-1} *$ Eq. (3) $\Rightarrow$

$$\left[[M]^{-1}[K] - \omega_n^2 [M]^{-1}[M]\right] \bar{X} = \bar{0}$$

or $$\boxed{[A - \omega_n^2 I]\, \bar{X} = \bar{0}} \quad (5)$$

Recall from Math 220:

$$[A - \lambda I]\bar{x} = \bar{0}$$

λ = eigenvalues of $[A]$ — n of them

$\bar{x}$ = eigenvectors of $[A]$

So, $\omega_n \sim \bar{X}$ is an eigenvalue-eigenvector problem:

ω_{ni}^2 = ith eigenvalue of system matrix $[A]$

$\bar{X}^{(i)}$ = ith eigenvector of $[A]$

Natural freqs and mode shapes bear a great importance in the design of high-speed machines:

$\omega_{n1}, \omega_{n2}, \ldots \Rightarrow$ critical speeds

$\bar{X}^{(1)}, \bar{X}^{(2)}, \ldots \Rightarrow$ rotor balancing.

Calculation of ω_n and $\bar{X}$

a) Hand calculation

use $\left[[K] - \omega_n^2 [M]\right] \bar{X} = \bar{0}$ (A)

b) Matlab calculation

use $[A - \lambda I] \bar{X} = \bar{0}$ (B) $[A] = [M]^{-1} [K]$

$\lambda = \omega_n^2$

Example:

Given

$$[M] = \begin{bmatrix} 11 & 0 \\ 0 & 22 \end{bmatrix}, \quad [K] = \begin{bmatrix} 1000 & -500 \\ -500 & 2000 \end{bmatrix}$$

Eq. (A) $\Rightarrow$

$$\begin{bmatrix} 1000 - 11\lambda & -500 \\ -500 & 2000 - 22\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1)$$

$$\text{Det}\left[[K] - \lambda[M]\right] = 0$$

$$\Rightarrow \quad (1000 - 11\lambda)(2000 - 22\lambda) - 500^2 = 0$$

$$\Rightarrow \quad \lambda_1 \text{ and } \lambda_2$$

then $\omega_{n1} = \sqrt{\lambda_1}$, $\omega_{n2} = \sqrt{\lambda_2}$

Eq. 1 $\Rightarrow$

$$\begin{bmatrix} 1000 - 11\lambda_1 & -500 \\ -500 & 2000 - 22\lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ X_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{X_2^{(1)} = \ldots}$$

and

$$\begin{bmatrix} 1000 - 11\lambda_2 & -500 \\ -500 & 2000 - 22\lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ X_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{X_2^{(2)} = \ldots}$$

```matlab
%An example of Programming in MATLAB to obtain
%natural frequencies and mode shapes of MDOF
%systems

%Define [M] and [K] matrices
M=[11 0;0 22]
K=[1000 -500;-500 2000]

%Form the system matrix
A=inv(M)*K

%Obtain eigenvalues and eigenvectors of A
[V,D]=eig(A)

%V and D above are matrices.
%V-matrix gives the eigenvectors and
%the diagonal of D-matrix gives the eigenvalues

% Sort eigen-values and eigen-vectors
[D_sorted, ind] = sort(diag(D),'ascend');
V_sorted = V(:,ind);

%Obtain natural frequencies and mode shapes
nat_freq_1 = sqrt(D_sorted(1))
nat_freq_2 = sqrt(D_sorted(2))
mode_shape_1 = V_sorted(:,1)
mode_shape_2 = V_sorted(:,2)
```

Results of MATLAB run

```
M =
    11     0
     0    22

K =
        1000        -500
        -500        2000

A =
   90.9091  -45.4545
  -22.7273   90.9091

V =
    0.8165    0.8165
   -0.5774    0.5774

D =
  123.0503         0
         0   58.7679

nat_freq_1 =
    7.6660

nat_freq_2 =
   11.0928

mode_shape_1 =
    0.8165
    0.5774

mode_shape_2 =
    0.8165
   -0.5774
```