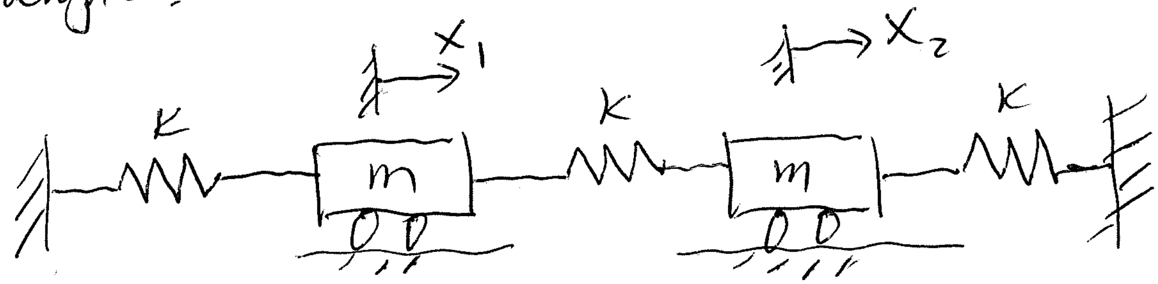
3.2 Free vibration of undamped MDOF system

Model:

$$[M]\ddot{\bar{x}} + [k]\bar{x} = \bar{0} \qquad (1)$$

Example:



Model Gov. eqs:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2)$$

ICs:

$$\bar{x}(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \qquad \dot{\bar{x}}(0) = \begin{bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{bmatrix}$$

$\bar{x}(t) = ?$ — must satisfy Eq. (2) and ICs.

Trial solution:

$$\bar{x}(t) = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega_n t + \phi) \qquad (3)$$

Verify:

a) satisfying Eq. (2)

$\Rightarrow$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} (-\omega_n^2) \sin(\omega_n t + \phi)$$

$$+ \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega_n t + \phi) \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Or

$$\begin{bmatrix} -m\omega_n^2 & 0 \\ 0 & -m\omega_n^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Or

$$\begin{bmatrix} (-m\omega_n^2 + 2k) & -k \\ -k & (-m\omega_n^2 + 2k) \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (4)$$

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} \neq \bar{0}$$

So, Eq. (4) can only be satisfied if and only if

$$\boxed{\text{Det} \begin{bmatrix} (-m\omega_n^2 + 2k) & -k \\ -k & (-m\omega_n^2 + 2k) \end{bmatrix} = 0} \qquad (5)$$

Eq. (5) = the characteristic eq. of the system.

Expanding the "Det" of Eq. (5)

$$\Rightarrow \quad (-m\omega_n^2 + 2k)^2 - k^2 = 0$$

Or $$\boxed{m^2\omega_n^4 - 4mk\,\omega_n^2 + 3k^2 = 0} \qquad (5)$$

Quadratic formula gives:

$$(\omega_n^2)_{1,2} = \frac{4mk \pm \sqrt{(4mk)^2 - 12m^2k^2}}{2m^2} = \frac{2k \pm k}{m}$$

So, two natural frequencies are:

$$\boxed{\omega_{n1} = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_{n2} = \sqrt{\frac{3k}{m}}}$$

and possible free vibration is:

$$\boxed{\bar{x}(t) = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^{(1)} \sin(\omega_{n1}t + \phi_1) + \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^{(2)} \sin(\omega_{n2}t + \phi_2)} \quad (6)$$

$\bar{x}(t)$ must satisfy ICs.

There are six underdetermined constants in Eq. 6:

$X_1^{(1)}$, $X_2^{(2)}$, ϕ_1, $X_1^{(2)}$, $X_2^{(2)}$ and ϕ_2.

They can't be uniquely determined by the four ICs

Two more eqs are needed before Eq. 6 can be verified to satisfy ICs.

consider part 1 alone in Eq. 6:

$$\bar{x}(t) = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(1)} \sin(\omega_{n1} t + \phi_1)$$

Eq. (4) $\Rightarrow$

$$\begin{bmatrix} (-m\overset{\frac{k}{m}}{\omega_{n1}^2} + 2k) & -k \\ -k & (-m\omega_{n1}^2 + 2k) \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad k\bar{X}_1^{(1)} - k\bar{X}_2^{(1)} = 0$$

$$-k\bar{X}_1^{(1)} + k\bar{X}_2^{(1)} = 0$$

$$\therefore \boxed{\bar{X}_1^{(1)} = \bar{X}_2^{(1)}} \quad \Rightarrow \quad \boxed{\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{X}^{(1)}}$$

Similarly, consider part 2 alone:

$$\bar{x}(t) = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(2)} \sin(\omega_{n2} t + \phi_2)$$

Eq. ⑥

$$\Rightarrow \begin{bmatrix} (-m\omega_{n2}^2 + 2k) & -k \\ -k & (-m\underbrace{\omega_{n2}^2}_{\frac{3k}{m}} + 2k) \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\bar{X}_1^{(2)} = -\bar{X}_2^{(2)}} \;\Rightarrow\; \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bar{X}^{(2)} \quad ⑧$$

Eqs. ⑦ and ⑧ into Eq. ⑥ ⇒

1st mode shape

2nd mode shape

$$\boxed{\bar{x}(t) = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{X}^{(1)} \sin(\omega_{n1} t + \phi_1)}_{\text{1st mode}} + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \bar{X}^{(2)} \sin(\omega_{n2} t + \phi_2)}_{\text{2nd mode}}} \quad ⑨$$

b) satisfying ICs

Eq. 9 $\Rightarrow$

$$\bar{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{X}^{(1)} \sin\phi_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bar{X}^{(2)} \sin\phi_2 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \quad (10)$$

and

$$\dot{\bar{x}}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{X}^{(1)} \omega_{n1} \cos\phi_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bar{X}^{(2)} \omega_{n2} \cos\phi_2 = \begin{bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{bmatrix} \quad (11)$$

Four eqs to determine four unknowns $\bar{X}^{(1)}, \bar{X}^{(2)}, \phi_1, \phi_2$

So, Eq. 9 is the solution.

Solving Eqs 10 and 11 to get the constants.

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Eq. (10.1) + Eq. (10.2) $\Rightarrow$

$$\bar{X}^{(1)} \sin\phi_1 = \frac{x_{10} + x_{20}}{2} \qquad (12)$$

Eq. (10.1) − Eq. (10.2) $\Rightarrow$

$$\bar{X}^{(2)} \sin\phi_2 = \frac{x_{10} - x_{20}}{2} \qquad (13)$$

Eq. (11.1) + Eq. (11.2) $\Rightarrow$

$$\bar{X}^{(1)} \cos\phi_1 = \frac{\dot{x}_{10} + \dot{x}_{20}}{2\omega_{n1}} \qquad (14)$$

Eq. (11.1) − Eq. (11.2) $\Rightarrow$

$$\bar{X}^{(2)} \cos\phi_2 = \frac{\dot{x}_{10} - \dot{x}_{20}}{2\omega_{n2}} \qquad (15)$$

$\text{Eq. } (12)^2 + \text{Eq. } (14)^2 \Rightarrow$

$$\boxed{X^{(1)} = \pm\left[\left(\frac{x_{10}+x_{20}}{2}\right)^2 + \left(\frac{\dot{x}_{10}+\dot{x}_{20}}{2\omega_{n1}}\right)^2\right]^{1/2}} \quad (16)$$

$\text{Eq. } (13)^2 + \text{Eq. } (15)^2 \Rightarrow$

$$\boxed{X^{(2)} = \pm\left[\left(\frac{x_{10}-x_{20}}{2}\right)^2 + \left(\frac{\dot{x}_{10}-\dot{x}_{20}}{2\omega_{n1}}\right]^{1/2}} \quad (17)$$

$\text{Eq. } (12) / \text{Eq. } (14) \Rightarrow$

$$\boxed{\phi_1 = \tan^{-1}_2 \frac{(x_{10}+x_{20})/2}{(\dot{x}_{10}+\dot{x}_{20})/2\omega_{n1}}} \quad (18)$$

$\text{Eq. } (13) / \text{Eq. } (15) \Rightarrow$

$$\boxed{\phi_2 = \tan^{-1}_2 \frac{(x_{10}-x_{20})/2}{(\dot{x}_{10}-\dot{x}_{20})/2\omega_{n2}}} \quad (19)$$

If ϕ_1 is well defined by Eq. (18) (i.e. not $\tan^{-1}\frac{0}{0}$), then Eq. (12) or Eq. (X) can be used to determine $\underline{X}^{(1)}$:

$$\boxed{\underline{X}^{(1)} = \frac{x_{10}+x_{20}}{2\sin\phi_1} \quad \text{or} \quad \frac{\dot{x}_{10}+\dot{x}_{20}}{2\omega_{n1}\cos\phi_1}} \tag{20}$$

Otherwise, Eq. (16) gives $\underline{X}^{(1)}=0$ and the 1st mode is absent from the free vibration (Eq. (9)). In this case, ϕ_1 does not need to be defined.

The above can be similarly applied to the solution of $\underline{X}^{(2)}$ if ϕ_2 is well defined:

$$\boxed{\underline{X}^{(2)} = \frac{x_{10}-x_{20}}{2\sin\phi_2} \quad \text{or} \quad \frac{\dot{x}_{10}-\dot{x}_{20}}{2\omega_{n2}\cos\phi_2}} \tag{21}$$

Otherwise, $\underline{X}^{(2)}=0$.