3.3 Harmonically excited vibration
(focus on follow-up responses)

Recall for an SDOF system:

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = E\sin\omega t \qquad (1)$$

Follow-up response is:

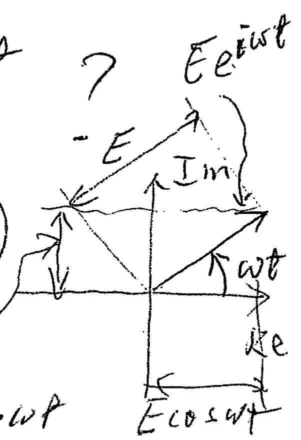
$$y(t) = Y\sin(\omega t + \psi) \qquad (2)$$

$$Y = \frac{E/\omega_n^2}{[(1-r^2)^2 + (2\xi r)^2]^{1/2}} \qquad (14)$$

$$\psi = -\tan^{-1}\frac{2\xi r}{1-r^2} \qquad (P)$$

What do we get for the follow-up response if the excitation is expressed by $Ee^{i\omega t}$?

Or $\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = Ee^{i\omega t} \qquad (3)$



$Ee^{i\omega t} = E\cos\omega t + iE\sin\omega t$

Trial solution =

$$y(t) = Y^* e^{i\omega t} \qquad (4) \qquad Y^* = \text{constant}$$

Eq (4) into Eq. (3)

$\Rightarrow$

$$Y^*(i\omega)^2 e^{i\omega t} + 2\zeta\omega_n Y^*(i\omega) e^{i\omega t} + \omega_n^2 Y^* e^{i\omega t} \stackrel{?}{=} E e^{i\omega t}$$

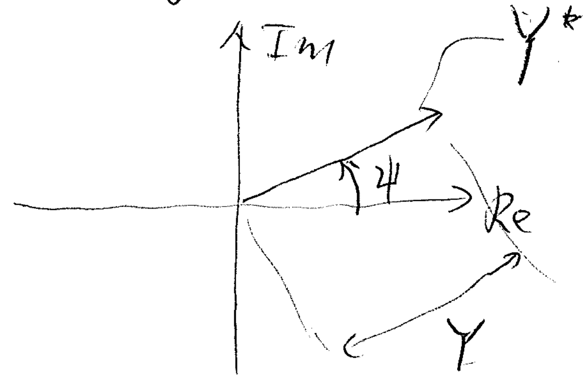
Or $$\left[(i\omega)^2 + 2\zeta\omega_n(i\omega) + \omega_n^2\right] Y^* e^{i\omega t} \stackrel{?}{=} E e^{i\omega t}$$

Or $$\left[(\omega_n^2 - \omega^2) + i(2\zeta\omega_n\omega)\right] Y^* \stackrel{?}{=} E \qquad \boxed{i^2 = -1.0}$$

$$\therefore \quad Y^* = \frac{E}{\left[(\omega_n^2-\omega^2) + i(2\zeta\omega_n\omega)\right]} = \frac{E/\omega_n^2}{\left[(1-r^2) + i(2\zeta r)\right]} \qquad (5)$$

So, Y^* in the trial solution is a complex number

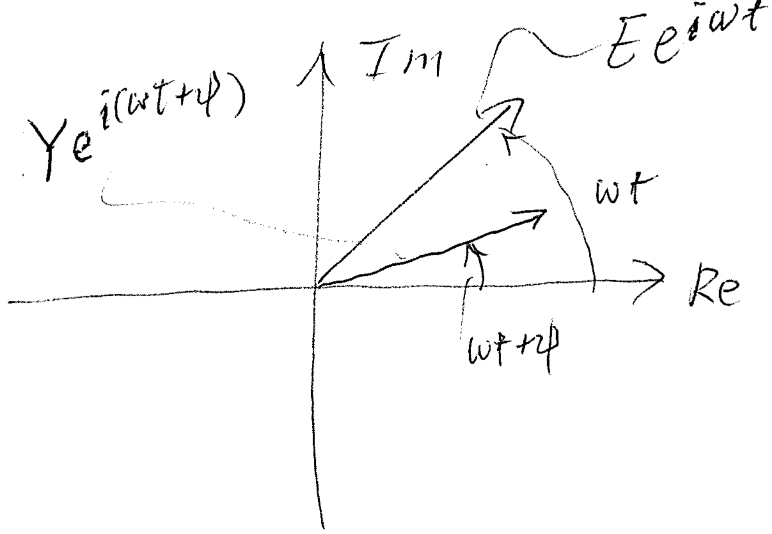
Or $$Y^* = Y e^{i\psi} \qquad (6)$$



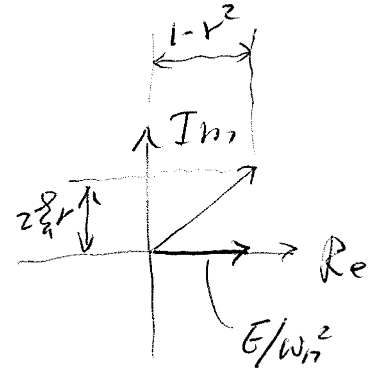
Eq. (4) $\Rightarrow$

$$y(t) = Y^* e^{i\omega t} = Y e^{i\phi} e^{i\omega t}$$

Or $$\boxed{y(t) = Y e^{i(\omega t + \phi)}} \qquad (7)$$



From the theory of complex numbers,



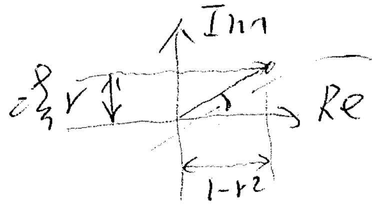
Eq. (5) $\Rightarrow$

$$Y = \frac{\text{Amp}\,(E/\omega_n^2)}{\text{Amp}\left[(1-r^2) + i(2\xi r)\right]}$$

Amp = amplitude

$$\therefore \quad Y = \frac{E/\omega_n^2}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{1/2}} \qquad (8)$$ — same as Eq. (M)

and



$\phi = \tan_2^{-1} \frac{Im}{Re}$

$$\psi = \text{angle}\,(E/\omega^2) - \text{angle}\left[(1-r^2)+i(2\xi r)\right]$$

angle = phase angle

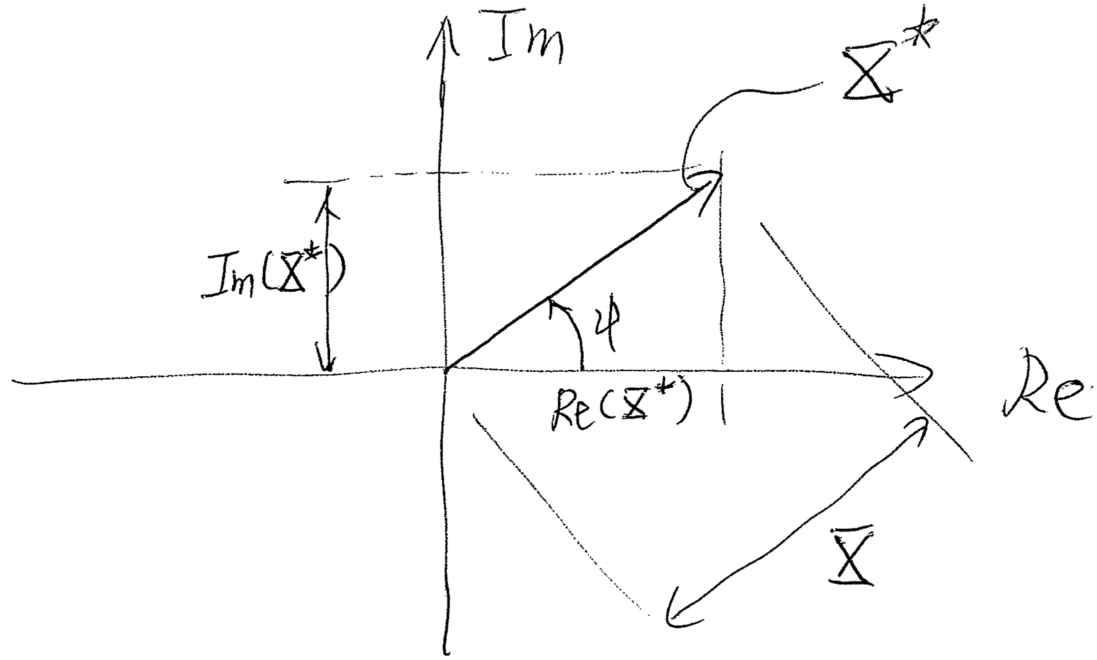
$$\therefore \boxed{\psi = 0 - \tan_2^{-1} \frac{2\xi r}{1-r^2}} \quad (9)$$

— same as Eq. (P)

So, the amplitude and phase lag ∴ the follow-up response obtained using $Ee^{i\omega t}$ excitation are the same as those using $E\sin\omega t$ or $E\cos\omega t$.

Relation between $(\bar{X}, \psi)$ and (Re, Im) of a complex number $\bar{X}^*$

$$\bar{X}^* = \bar{X} e^{i\psi} = Re(\bar{X}^*) + i\, Im(\bar{X}^*)$$



$\therefore$

$$\boxed{\begin{aligned} Re(\bar{X}^*) &= \bar{X} \cos\psi \\ Im(\bar{X}^*) &= \bar{X} \sin\psi \end{aligned}}$$

and

$$\boxed{\begin{aligned} \bar{X} &= \left[Re^2(\bar{X}^*) + Im^2(\bar{X}^*)\right]^{1/2} \\ \psi &= \tan^{-1} \frac{Im(\bar{X}^*)}{Re(\bar{X}^*)} \end{aligned}}$$