Review – Cross-Integration for a Function of Two Variables

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Introduction and Review: Total Integration for a function of one variable

Total integration is straightforward; when we integrate we add a constant of integration. Let's illustrate with a simple example:

Given: $g = g(x) \qquad \frac{dg}{dx} = 3x$

To do: Write a general expression for function g.

Solution:

We integrate g with respect to x. Note that this is a ***total integration*** since g is a function of x only.

$$\frac{dg}{dx} = 3x \quad \rightarrow \quad g = \frac{3x^2}{2} + C \tag{1}$$

where C is some arbitrary constant.

We check our answer by differentiating Eq. (1): $g = \frac{3x^2}{2} + C \quad \rightarrow \quad \frac{dg}{dx} = \frac{2 \cdot 3x}{2} = 3x$. We are confident now that our result is correct. Our final answer is thus $\boxed{g(x) = \frac{3x^2}{2} + C}$.

Cross-Integration for a function of two variables

When we have a function of *two* variables, total integration is not so straightforward because we have to ***partially integrate*** and then add not just a constant but a function of the other variable. Let's again illustrate with a simple example:

Given: $g = g(x, y) \qquad \frac{\partial g}{\partial x} = 3y \qquad \frac{\partial g}{\partial y} = 3x + 1$

Note that g is now a function of both x and y and so we have expressed the derivatives as ***partial derivatives*** (symbol ∂) instead of ***total derivatives*** (symbol d).

To do: Generate a general expression for function g.

Solution:

We start with either of the two partial derivative equations and integrate. Note that this is a ***partial integration*** since g is a function of both x and y and so we have to add a function of the other variable instead of just a constant when we integrate:

$$\frac{\partial g}{\partial x} = 3y \quad \rightarrow \quad g = 3xy + f(y) \tag{2}$$

where $f(y)$ is some arbitrary function of y. Note that here, since we partially integrated with respect to x, we added a function of the other variable (y) instead of a constant. We can verify this step by differentiating Eq. (2) with respect to x:

$g = 3xy + f(y) \quad \rightarrow \quad \frac{\partial g}{\partial x} = 3y + 0 = 3y$. We see that this holds for any arbitrary function $f(y)$ since $\partial f(y)/\partial x = 0$.

To continue, we differentiate our expression for g from Eq. (2) with respect to the *other* variable, in this case y:

$$g = 3xy + f(y) \quad \rightarrow \quad \frac{\partial g}{\partial y} = 3x + \frac{df}{dy}$$

Now we need to equate this result to the original given expression for the partial derivative of g with respect to y:

$$\frac{\partial g}{\partial y} = 3x + \frac{df}{dy} = 3x + 1 \quad \rightarrow \quad \frac{df}{dy} = 1 \quad \rightarrow \quad f(y) = y + C \tag{3}$$

Finally, we substitute Eq. (3) into Eq. (2) to get our final answer:

$$g = 3xy + f(y) = 3xy + y + C \quad \rightarrow \quad g(x, y) = 3xy + y + C$$

We check our answer by differentiating with respect to each variable: $\frac{\partial g}{\partial x} = 3y$ and $\frac{\partial g}{\partial y} = 3x + 1$ which agree with the given expressions. We are confident now that our result is correct. Our final answer is thus $\boxed{g(x, y) = 3xy + y + C}$.