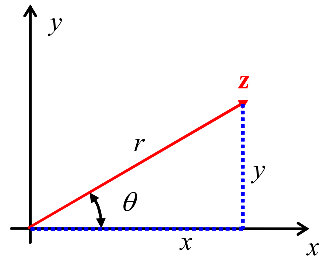
Review of Complex Variables

Author: John M. Cimbala, Penn State University
Latest revision: 23 October 2007

Basic Definitions in the Complex Plane

Define $\boxed{z \equiv x + iy = re^{i\theta}}$, where x and y are *real* numbers, and z is a *complex* number made up of a real part (x) and an imaginary part (iy). The imaginary unity number is $\boxed{i \equiv \sqrt{-1}}$. Complex number z is often represented graphically on the complex x-y plane as shown in the sketch. Complex number z can also be written in terms of r and θ, where r is the magnitude of z, $\boxed{r = |z| = \sqrt{x^2 + y^2}}$ (often thought of as a "radius"), and θ is the angle between the x axis and the ray z as shown on the sketch. Mathematically, $\boxed{\theta \equiv \arctan(y/x)}$.



The complex x-y plane

Some Rules and Review

a) Complex Conjugate

The ***complex conjugate*** of a complex variable $z = x + iy$ is obtained by changing the sign of the imaginary part of z. Namely, the complex conjugate of z is defined as $\boxed{z^* \equiv x - iy = re^{-i\theta}}$. (Everywhere an i appears, change it to a $-i$.)

b) Magnitude of a Complex Variable

The ***magnitude*** (also called the ***modulus***) of a complex variable $z = x + iy$ is obtained by taking the square root of the product of z and its complex conjugate, i.e., $\boxed{|z| = \sqrt{zz^*}}$. Note that this can be expanded to give $\boxed{|z| = r}$.

c) Miscellaneous Equations

$\boxed{\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{array}}$ $\boxed{\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}}$ $\boxed{\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}}$ $\boxed{\text{For } z_1 = r_1 e^{i\theta_1} \text{ and } z_2 = r_2 e^{i\theta_2},\ z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}}$

d) Separating a Complex Function into Real and Imaginary Parts

A "trick" to separate a complex function into its real and imaginary parts is to multiply and divide by the complex conjugate of the denominator of the function. (This always works because it guarantees that the denominator will become real.)

Example: Calculate the real and imaginary parts of $F(z) = c/z$, where c is a real constant.

Solution: $F = \dfrac{c}{x+iy} = \dfrac{c}{x+iy}\dfrac{(x-iy)}{(x-iy)} = \dfrac{cx}{x^2+y^2} - \dfrac{icy}{x^2+y^2}$.

So, $\boxed{\text{the real part is } \dfrac{cx}{x^2+y^2}}$, and $\boxed{\text{the imaginary part is } \dfrac{-cy}{x^2+y^2}}$.

e) Derivatives of Complex Functions

Differentiation of complex functions is relatively simple because it follows the same basic rules as does differentiation of real functions (product rule, exponent rules, etc.). *Note*: **F(z) is an *analytic function* in any region where a finite, unique derivative, *dF/dz*, can be defined everywhere within the region**. If $F(z)$ is an analytic function, then dF/dz is independent of the orientation of dz in the x-y plane.

Examples:

	$F(z) = z^2$	$\dfrac{dF}{dz} = 2z$	(exponent rule)
	$F(z) = \ln(z)$	$\dfrac{dF}{dz} = \dfrac{1}{z}$	(natural log rule)
	$F(z) = \dfrac{\ln(z)}{z}$	$\dfrac{dF}{dz} = \dfrac{1}{z}\cdot\dfrac{1}{z} + \ln(z)\cdot\left(-\dfrac{1}{z^2}\right)$	(product rule combined with above rules)