EQUATION OF FLUID STATICS

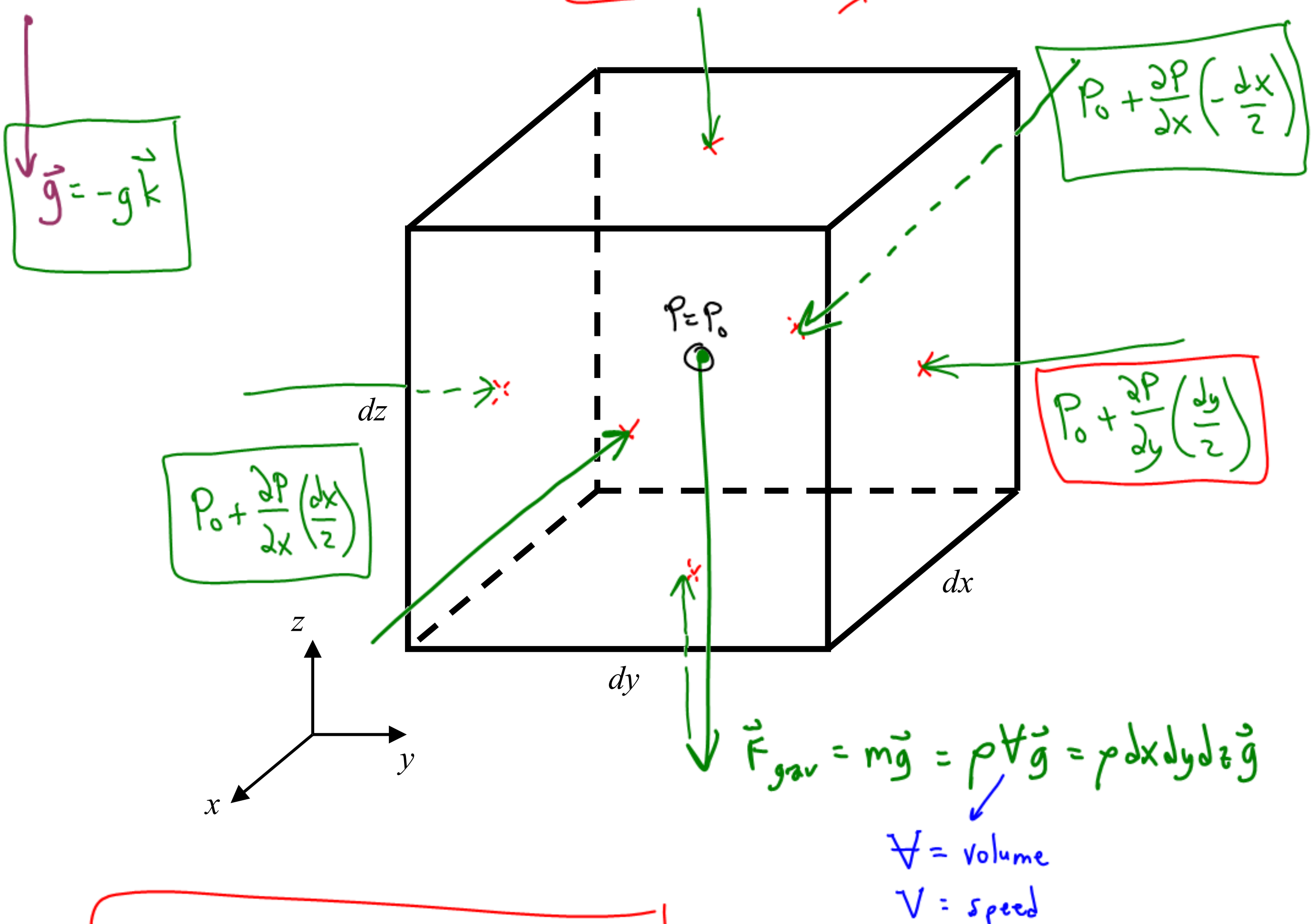
In this lesson, we will:

- Derive the **equation of fluid statics**
- Discuss a simplified case for an incompressible fluid

Derivation – The Equation of Fluid Statics

Consider an infinitesimal fluid element of dimensions *dx*, *dy*, and *dz* as sketched below.

Hydrostatics $\rightarrow \boxed{\sum \vec{F} = 0} \rightarrow \sum \vec{F} = \sum \vec{F}_{\text{body forces}} + \sum \vec{F}_{\text{surface forces}} = 0$



$$\boxed{\sum \vec{F}_{\text{body forces}} = -\rho g\, dx\, dy\, dz\, \vec{k}} \;*$$

Pressure – Consider the average pressure on each face

* Pressure acts inward & normal at any surface

$P = P(x, y, z, t)$ Thus, we use partial derivatives ∂, not total derivatives d

☆ We use truncated Taylor series expansions

Consider a point ds away from the center of our fluid element

$$\boxed{P(\text{new location}) = P_0 + \frac{\partial P}{\partial s} ds} + \cancel{\frac{1}{2!}\frac{\partial^2 P}{\partial s^2} ds^2} + \dots$$

$ds \to 0$

Ignore higher order terms

Sum all surface forces in the x-direction:

$$\sum F_{\text{surface},x} = \underbrace{-\left(\cancel{P_0} + \frac{\partial P}{\partial x}\frac{dx}{2}\right)dy\,dz}_{\text{Force on front face}} + \underbrace{\left(\cancel{P_0} - \frac{\partial P}{\partial x}\frac{dx}{2}\right)dy\,dz}_{\text{Force on back face}}$$

$$\boxed{\sum F_{\text{surface},x} = -\frac{\partial P}{\partial x}\,dx\,dy\,dz}$$ ☆

Similarly for y & z directions,

$$\boxed{\begin{aligned}\sum F_{\text{surface},y} &= -\frac{\partial P}{\partial y}\,dx\,dy\,dz\\ \sum F_{\text{surface},z} &= -\frac{\partial P}{\partial z}\,dx\,dy\,dz\end{aligned}}$$

☆ Hydrostatic Equation $\qquad \sum \vec{F} = 0$

$$\sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}} = 0$$

Pressure does not vary in the x-direction in hydrostatics

Consider the x-direction ($\hat{i}$)

$$0 + -\frac{\partial P}{\partial x}\,dx\,dy\,dz = 0 \Rightarrow \boxed{\frac{\partial P}{\partial x} = 0}$$ ☆

$\downarrow \vec{g} \qquad \uparrow z$

- Similarly, y-direction,

$$0 - \frac{\partial P}{\partial y}\,dx\,dy\,dz = 0 \Rightarrow \boxed{\frac{\partial P}{\partial y} = 0} ☆$$

☆ P does not vary in the y-direction

- z-direction

$$\underbrace{-\rho g\, dx\,\cancel{dy}\,dz - \frac{\partial P}{\partial z}\,dx\,\cancel{dy}\,dz}_{\text{cancel}} = 0$$

$$\therefore \boxed{\frac{\partial P}{\partial z} = -\rho g} ☆ \qquad (1)$$

☆ Pressure does vary in the z-direction in hydrostatics

Bottom line: In fluid statics, in a continuous fluid,

☆ P does not vary horizontally
but P does vary vertically ☆

Due to the negative sign in Eq. (1), P decreases as you go up
or P increases as you go down

☆

recall, $\boxed{P_{below} = P_{above} + \rho g|\Delta z|}$ agrees ✓

WE CAN SOLVE ANY HYDROSTATICS PROBLEM WITH THIS

Hydrostatics or Fluid Statics

liquids — liquids or gases

$$P = P(\cancel{x}, \cancel{y}, z, \cancel{t}) \rightarrow \boxed{P = P(z) \text{ only in fluid statics}}$$

So, Eq. (1) $\frac{\partial P}{\partial z} = -\rho g$ $\Rightarrow$ we can use d instead of ∂

EQ. OF FLUID STATICS ☆ → $\boxed{\frac{dP}{dz} = -\rho g}$ ☆ (2)

INTEGRATE TO GET P_2 FOR A KNOWN P_1

$$dP = -\rho g \, dz$$

$$\boxed{\int_{P_1}^{P_2} dP = -\int_{z_1}^{z_2} (\rho g) \, dz}$$ ☆ General case

Simplest case : Incompressible case $\rightarrow$ ρ = constant g is constant unless Δz is huge

☆ $$\boxed{P_2 - P_1 = -\rho g \int_{z_1}^{z_2} dz = -\rho g (z_2 - z_1)}$$

Holds for any incompressible fluid at rest

can rewrite as our simple eq. from previously