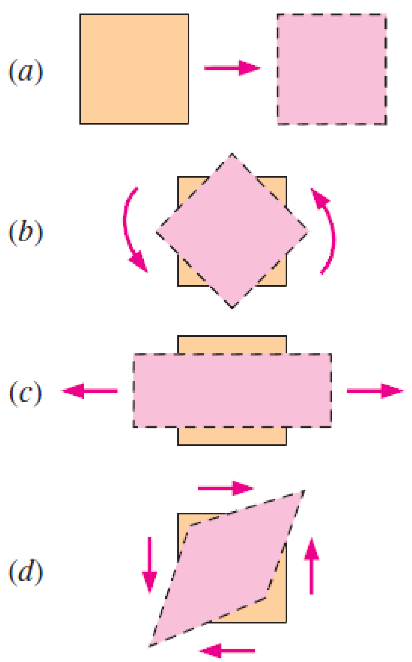
## TRANSLATION, ROTATION, AND VORTICITY

**In this lesson, we will**:

- Discuss the fundamental types of fluid element motion or deformation
- Define vorticity and how it is related to rotationality
- Do some example problems

### The Four Fundamental Types of Fluid Element Motion or Deformation




(a) TRANSLATION (Fluid element moves)

(b) ROTATION (Fluid element rotates)

NO DEFORMATION (Just motion)

(c) LINEAR STRAIN (Fluid element stretches)

(d) SHEAR STRAIN (Fluid element shears)

DEFORMATION

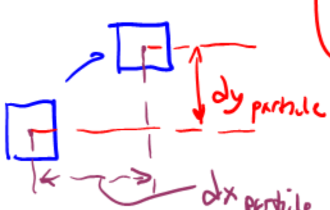
☆ All four of these can occur simultaneously in a fluid flow

☆ In fluid mechanics, we prefer to use rates of motion or deformation

### Rate of Translation

$$= \vec{V} = \frac{dx_{particle}}{dt}\vec{i} + \frac{dy_{particle}}{dt}\vec{j} + \frac{dz_{particle}}{dt}\vec{k}$$

$$\boxed{\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}}$$ ☆




$$\{\vec{V}\} = \left\{\frac{L}{t}\right\} \qquad [\vec{V}] = \left[\frac{m}{s}\right]$$

### Rate of Rotation

In Cartesian coordinates, the **rate of rotation** of a fluid element is $\{\vec{\omega}\} = \left\{\frac{L}{t}\frac{1}{L}\right\} = \left\{\frac{1}{t}\right\}$

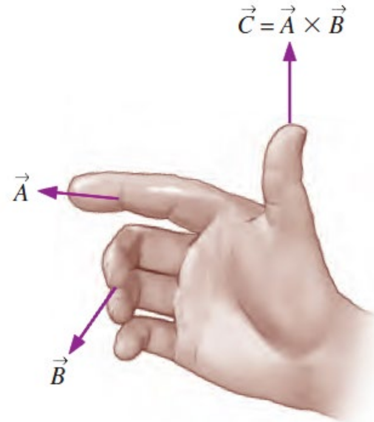
$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$ = angular velocity vector

$$[\omega] = \left[\frac{1}{s}\right]$$

## Vorticity

The **vorticity vector** is defined as the **curl of the velocity vector**, using the **right-hand rule**.

Greek letter zeta → $\vec{\zeta} = \vec{\nabla} \times \vec{V}$




It turns out that **vorticity is equal to twice the angular velocity of a fluid particle**,

$$\vec{\zeta} = 2\vec{\omega}$$

$\{\vec{\zeta}\} = \{\frac{1}{t}\}$ $[\zeta] = [\frac{1}{s}]$

Thus, ***vorticity is a measure of rotation of a fluid particle***.

Some authors use $\vec{\omega}$ for vorticity

if $\vec{\zeta} = 0$, the flow is irrotational

if $\vec{\zeta} \neq 0$, the flow is rotational

The vorticity vector in Cartesian coordinates:

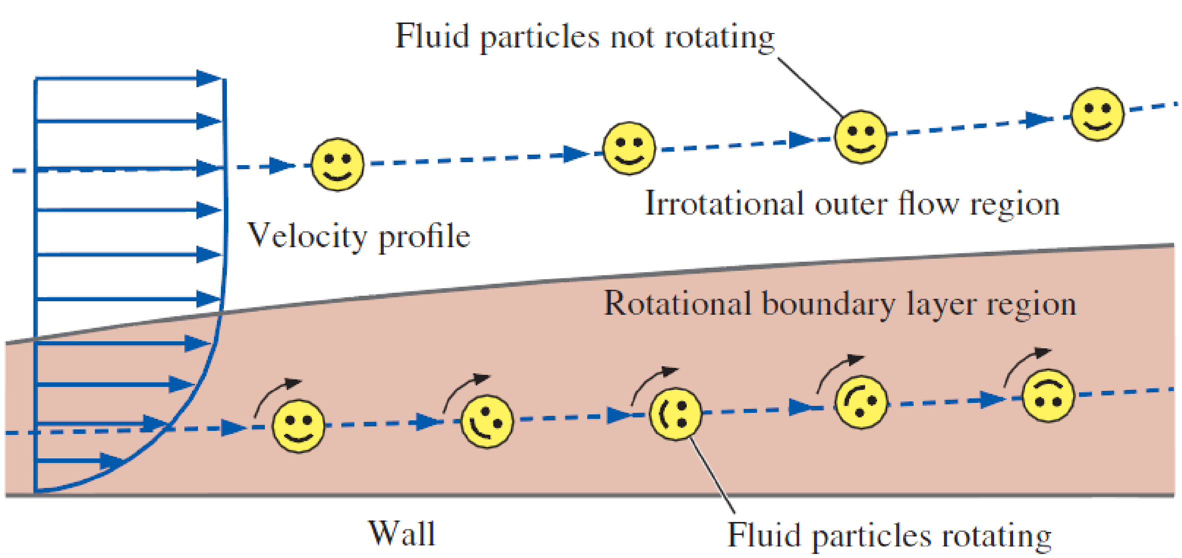
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

The vorticity vector in cylindrical coordinates: $(r, \theta, z)$ $(u_r, u_\theta, u_z)$

$$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z$$

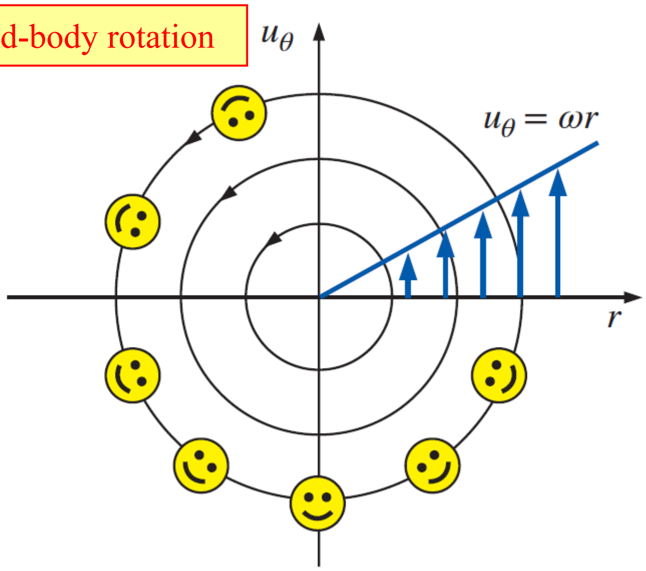
**Examples**:

1. Inside a **boundary layer**, where viscous forces are important, the flow in this region is ***rotational*** ($\vec{\zeta} \neq 0$). However, outside the boundary layer, where viscous forces are not important, the flow in this region is ***irrotational*** ($\vec{\zeta} = 0$).



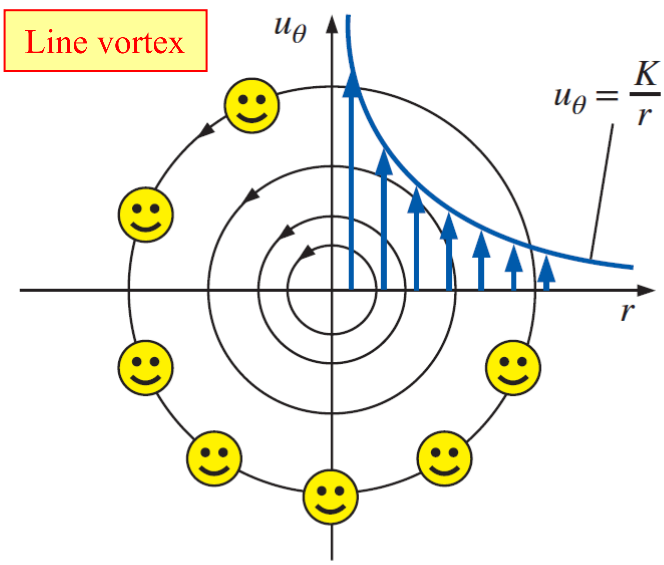
2. Flow in **solid-body rotation** (rigid-body rotation) is ***rotational*** ($\vec{\zeta} \neq 0$). In fact, since vorticity is equal to twice the angular velocity, $\vec{\zeta} = 2\vec{\omega}$ ***everywhere*** in the flow field. Fluid particles rotate as they revolve around the center of the flow. This is analogous to a merry-go-round or a roundabout.

Solid-body rotation




3. A **line vortex** flow, however, is ***irrotational*** ($\vec{\zeta} = 0$), and fluid particles do not rotate, even though they revolve around the center of the flow. This is analogous to a Ferris wheel.

Line vortex




**Demo by Duck Dynamics and Dick Dynamics**

Visual aids created by former students **Caitlin Hensley** and **Morgan Austin**.

## Example: Vorticity and irrotationality

**Given**: A two-dimensional velocity field in the *x*-*y* plane: $\boxed{\vec{V} = (u,v) = 2xy\vec{i} - y^2\vec{j}}$. $(w = 0)$

**To do**: Is this flow rotational or irrotational? (Steady)

$u = 2xy \qquad v = -y^2$

**Solution**:

The rate of rotation is

$$\vec{\omega} = \frac{1}{2}\left(\cancel{\frac{\partial w}{\partial y}} - \cancel{\frac{\partial v}{\partial z}}\right)\vec{i} + \frac{1}{2}\left(\cancel{\frac{\partial u}{\partial z}} - \cancel{\frac{\partial w}{\partial x}}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$ and vorticity = $\vec{\zeta} = 2\vec{\omega}$.

0 (2-D) 0 (2-D) 0 (2-D) 0 (2-D) 0 ($v \neq$ fnc of $x$) $\frac{\partial}{\partial y}(2xy) = 2x$

$$\vec{\omega} = 0\vec{i} + 0\vec{j} - x\vec{k} \Rightarrow \boxed{\begin{aligned}\vec{\omega} &= -x\vec{k} \\ \vec{\zeta} &= -2x\vec{k}\end{aligned}}$$

Since $\vec{\zeta} \neq 0$, THIS FLOW IS ROTATIONAL

## Example: Vorticity and irrotationality

**Given**: A two-dimensional velocity field in the *x*-*y* plane: $\boxed{\vec{V} = (u,v) = 3x\vec{i} - 3y\vec{j}}$. $(w = 0)$

**To do**: Calculate (**a**) the rate of translation and (**b**) the rate of rotation.

**Solution**:

(**a**) The rate of translation is simply the velocity vector,

$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$

$$\boxed{\vec{V} = 3x\vec{i} - 3y\vec{j}} \rightarrow \text{or} \quad \boxed{\begin{aligned}u &= 3x \\ v &= -3y \\ w &= 0\end{aligned}}$$

(**b**) The rate of rotation is

$$\vec{\omega} = \frac{1}{2}\left(\cancel{\frac{\partial w}{\partial y}} - \cancel{\frac{\partial v}{\partial z}}\right)\vec{i} + \frac{1}{2}\left(\cancel{\frac{\partial u}{\partial z}} - \cancel{\frac{\partial w}{\partial x}}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$ and vorticity = $\vec{\zeta} = 2\vec{\omega}$.

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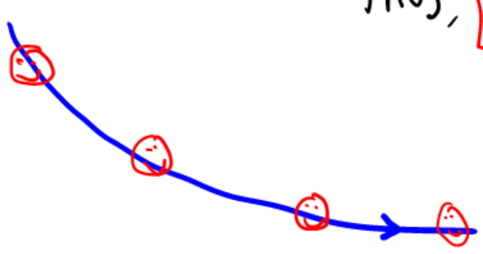
$\rho$ = rho

$\zeta$ = zeta

THUS, $\boxed{\vec{\omega} = 0}$ & $\boxed{\vec{\zeta} = 0}$



THIS FLOW IS IRROTATIONAL