CONSERVATION OF MASS

In this lesson, we will:

- Use the Reynolds Transport Theorem to generate the **equation of conservation of mass for a control volume** and show some simplifications of this equation
- Discuss **mass flow rate** and **volume flow rate**
- Do some example problems

Derivation of Conservation of Mass from the RTT

For a system

$$\boxed{\frac{dm_{sys}}{dt} = 0}\ \star$$

Apply RTT: consider a fixed CV for simplicity

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, d\forall + \int_{CS} \rho b (\vec{V}\cdot\vec{n})\, dA$$

Let $B = m = \text{mass} \rightarrow b = \frac{B}{m} = \frac{m}{m} = 1$

$$\frac{dm_{sys}}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho\, d\forall + \int_{CS} \rho (\vec{V}\cdot\vec{n})\, dA \qquad (1)$$

CONS. OF MASS FOR A SYSTEM

CONS. OF MASS EQ FOR A FIXED CV $\star$

rate of change of mass within the CV $= \frac{d}{dt} m_{CV}$

net rate of mass flow out of the system through the CS

If $\oplus$ — must be $\ominus$

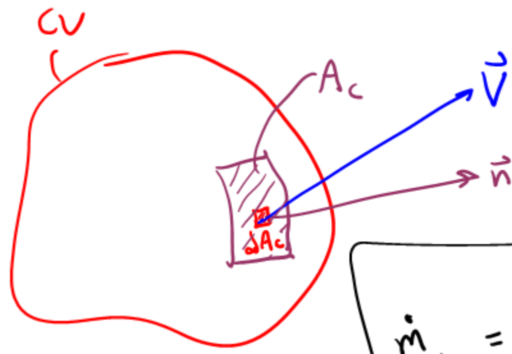
If $\ominus$ — must be $\oplus$

If 0 — must be 0

STEADY FLOW

For steady flow, $\star$ there is no net mass flow out of the CV

Mass Flow Rate



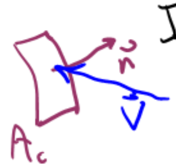
$$\{\dot{m}\} = \left\{\frac{\text{mass}}{\text{time}}\right\}$$

$$\boxed{\dot{m}_{A_c} = \int_{A_c} \rho\left(\vec{V}\cdot\vec{n}\right)dA_c}$$

$\dot{m}_{A_c}$ = mass flow rate <u>out</u> of area A_c

If mass flow rate is <u>out</u>, $\vec{V}\cdot\vec{n} > 0$, $\therefore$ $\underline{\underline{\dot{m}_{A_c} > 0}}$

★ THIS IS AN <u>OUTLET</u>



If mass flow rate is <u>in</u>, $\underline{\vec{V}\cdot\vec{n} < 0}$ $\therefore$ $\underline{\underline{\dot{m}_{A_c} < 0}}$

★ THIS IS AN <u>INLET</u>

If A_c = entire CS, then

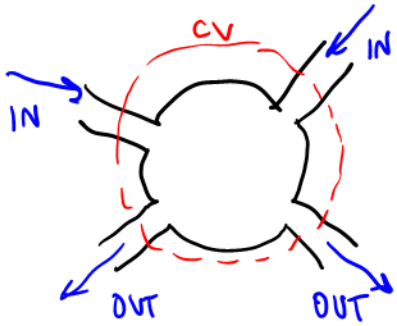
$$\boxed{\dot{m}_{cs} = \int_{cs} \rho\left(\vec{V}\cdot\vec{n}\right)dA} \qquad (2)$$

2^{nd} term in our cons of mass eq. for a CV (1)

THIS IS THE NET RATE OF MASS FLOW OUT OF THE CS ★

• <u>Simplification</u>: We usually have well defined inlets & outlets.

So, we can split this integral of Eq. (2) into several parts, one for each inlet & outlet



let $\boxed{\dot{m}_{cs} = \sum_{out} \dot{m} - \sum_{in} \dot{m}}$

where $\dot{m}$ is $\oplus$ for both inlets & outlets

Eq (1) becomes $\frac{d}{dt} \underbrace{\int_{CV} \rho \, d\forall}_{m_{CV}} = \sum_{in} \dot{m} - \sum_{out} \dot{m}$

or $\boxed{\frac{d}{dt} m_{CV} = \sum_{in} \dot{m} - \sum_{out} \dot{m}}$

rate of change of mass within the CV = net rate of mass flow into the CV

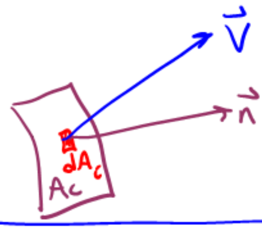
<u>Special simple case</u> for <u>Steady Flow</u>

$\frac{d}{dt} m_{CV} = 0$

$\therefore \boxed{\sum_{in} \dot{m} = \sum_{out} \dot{m}}$ ★

STEADY FLOW CONS. OF MASS FOR A FIXED CV ★

Volume Flow Rate



Define $\boxed{\dot{\forall}_{A_c} = \int_{A_c} (\vec{V}\cdot\vec{n})\, dA_c}$ = volume flow rate <u>out</u> of area A_c

Notation: $\forall$ = volume, $\dot{\forall}$ = volume flow rate
[m = mass $\dot{m}$ = mass flow rate]
Some other authors use <u>Q</u> for volume flow rate

Average velocity (actually speed) through A_c

$$V_{avg,A_c} = \frac{1}{A_c} \underbrace{\int_{A_c} (\vec{V}\cdot\vec{n})\, dA_c}_{= \dot{\forall}_{A_c}}$$

$$\boxed{V_{avg,A_c} = \frac{\dot{\forall}_{A_c}}{A_c}}$$

<u>Approximation</u> for a given inlet or outlet

$$\boxed{\dot{m}_{A_c} \approx \rho_{avg} \cdot V_{avg} A_c}$$

or $\boxed{\dot{m}_{A_c} \approx \rho V A_c} = \rho \dot{\forall}_{A_c}$ $\boxed{\dot{m}_{A_c} = \rho \dot{\forall}_{A_c}}$

- For incompressible flow $(\rho \approx \text{const})$

and steady, with well defined inlets & outlets

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

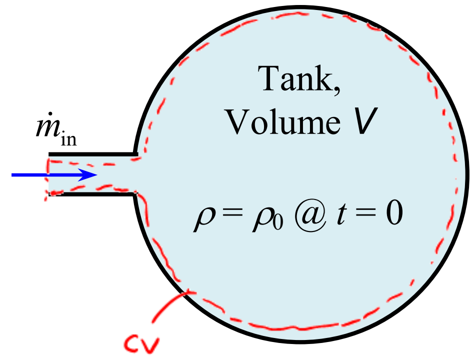
$$\sum_{in} \left[\cancel{\rho} \dot{\forall}\right] = \sum_{out} \left[\cancel{\rho} \dot{\forall}\right]$$

Incomp

$$\boxed{\sum_{in} \dot{\forall} = \sum_{out} \dot{\forall}}$$

Example: Unsteady conservation of mass (flow into a tank)

Given: Air is pumped into a rigid tank of volume V. The mass flow rate of the air entering the tank is constant, $\dot{m}_{in}$. We assume that the process is slow enough that the air in the tank remains at the same temperature (isothermal conditions).



To do: Generate an equation for density ρ in the tank as a function of time.

Solution:

- Draw an appropriate C.V.

- Cons. of mass eq.

$$\frac{d}{dt}\int_{CV} \rho \, d\forall = \sum_{in} \dot{m} - \cancel{\sum_{out} \dot{m}}$$

no outlet

- Assume $\rho = \rho(t)$ only

$$\frac{d\rho}{dt}\underbrace{\int_{CV} d\forall}_{\forall} = \dot{m}_{in}$$

($\forall$ is constant since the tank is rigid)

$$\frac{d\rho}{dt} = \frac{\dot{m}_{in}}{\forall} = \text{constant}$$

- Integrate from $t=0$ where $\rho = \rho_0$ to some later time t

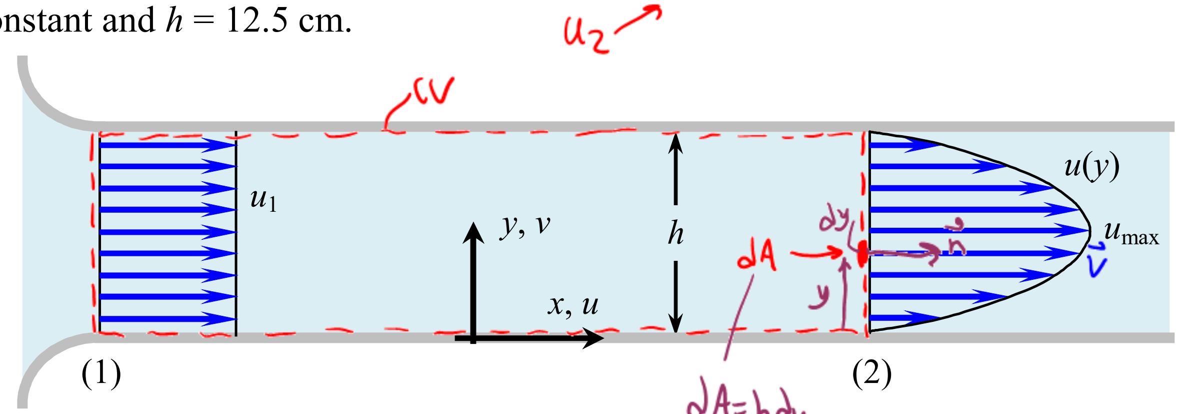
$$\int_{\rho_0}^{\rho} d\rho = \frac{\dot{m}_{in}}{\forall}\int_{t=0}^{t} dt$$

$$\rho - \rho_0 = \frac{\dot{m}_{in}}{\forall}(t-0) \longrightarrow \boxed{\rho = \rho_0 + \frac{\dot{m}_{in}}{\forall}t}$$

Example: Velocity profiles in 2-D channel flow

Given: Consider steady, incompressible, two-dimensional flow of a liquid between two very long parallel plates as sketched. At the inlet (1) there is a nice bell mouth, and the velocity is nearly uniform (except for a very thin, negligible boundary layer, not shown).

- At (1), $u = u_1$ = constant = 3.00 m/s, $v = 0$, and $w = 0$.
- At (2), the flow is fully developed, and $u = ay(h - y)$, $v = 0$, and $w = 0$, where a is a constant and h = 12.5 cm.



To do: Calculate constant a and speed u_{max}.

Solution:

• Pick a wise control volume ★

• Cons. of mass   Here — • steady • incompressible $\Big\}$ $\sum_{in} \dot{V} = \sum_{out} \dot{V}$

only one ①   only one ②

[b = width into the page]

$$u_1 b h = \int_{A_2} \vec{V}\cdot\vec{n}\, dA \qquad dA = b\,dy$$

$$\vec{V}\cdot\vec{n} = (u_2, 0, 0)\cdot(1, 0, 0) = u_2$$

$$u_1 \cancel{b} h = \int_{y=0}^{h} u_2 \cancel{b}\, dy$$

$$u_1 h = \int_{y=0}^{y=h} ay(h-y)\, dy$$

$$u_1 h = \frac{ah^3}{6} \rightarrow \boxed{a = \frac{6u_1}{h^2}} = \frac{6(3.00\ \text{m/s})}{(0.125\ \text{m})^2} = \boxed{1152\ \frac{1}{\text{m s}} = a}$$

u_{max} occurs @ $y = h/2$ $\longrightarrow$ $u_{max} = a\frac{h}{2}\left(h - \frac{h}{2}\right) = a\frac{h^2}{4}$

or $\boxed{u_{max} = \frac{3u_1}{2}}$

$$\boxed{\begin{array}{l} a = 1150\ \frac{1}{\text{ms}} \\ u_{max} = 4.50\ \text{m/s} \end{array}}$$