LINEAR MOMENTUM, MOVING CONTROL VOLUME

In this lesson, we will:

- Discuss the linear momentum equation for a moving control volume
- Do some example problems

Linear Momentum Equation for a Moving Control Volume

We derived the linear momentum equation for a fixed control volume in the previous lesson, using the Reynolds Transport Theorem. For a moving or deforming control volume, the main difference involves using relative velocity instead of actual velocity.

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \int_{CS} \rho \vec{V}\left(\vec{V}_r \cdot \vec{n}\right) dA,$$

Total force (***vector***) acting on the control volume

Rate of change of linear momentum inside the control volume

Net rate of linear momentum flow out of the control volume

For a general moving or deforming control volume, we use *relative* velocity $\vec{V}_r$ in this term only, not in the other term in this integral

For a *non-accelerating* and *non-distorting* control volume, we also use *relative* velocity $\vec{V}_r$ in the other term in this integral

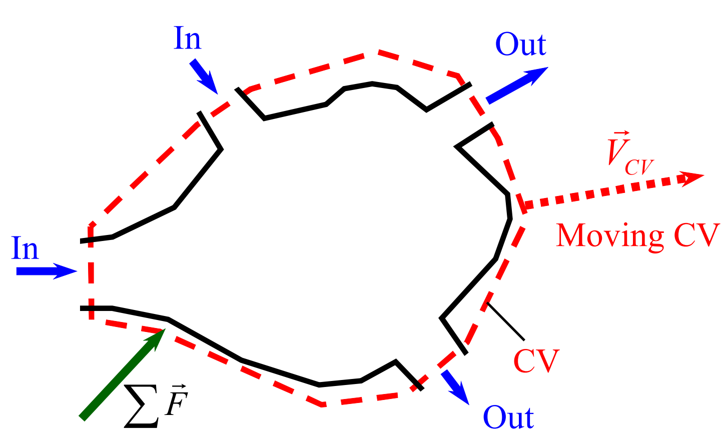
Simplifications, Assumptions, and Approximations:

- Assume well-defined inlets and outlets
- Apply momentum flux correction factors
- Split total force into gravity, pressure, viscous, and other forces as previously
- Consider only an ***inertial reference frame*** for simplicity → NOT ACCELERATING☆

Approximate, most useful form of the linear momentum equation:

$$\boxed{\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m}_r \vec{V}_r - \sum_{\text{in}} \beta \dot{m}_r \vec{V}_r}$$

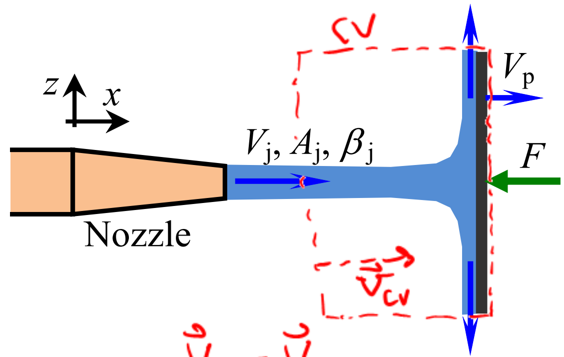
This equation is valid for a fixed-shape control volume with well-defined inlets and outlets moving at a constant speed (inertial reference frame).



Examples

Example: Force imparted by a water jet hitting a *moving* plate

Given: A horizontal water jet of area A_j, average velocity V_j, and momentum flux correction factor β_j impinges normal to a *moving* vertical flat plate. The plate moves to the right at constant speed V_p.



$\vec{V}_{cv} = \vec{V}_p$

To do: Calculate the horizontal force F required to keep the plate moving at constant speed V_p.

Solution:

- First step: Pick a wise CV
- Second step: Use the approximate, most useful form of the linear momentum equation, in the x-direction, for a moving CV, but *steady*:

$$\boxed{\sum F_x = \sum \cancel{F_{x,\text{ gravity}}} + \sum \cancel{F_{x,\text{ pressure}}} + \sum \cancel{F_{x,\text{ viscous}}} + \sum \underbrace{F_{x,\text{ other}}}_{= -F} = \sum_{\text{out}} \beta \dot{m}_r \cancel{u_r} - \sum_{\text{in}} \beta \dot{m}_r u_r}$$

none in x   P = P_atm everywhere   wise CV   0 @ outlet

What is u_r? $u_r = u_{absolute} - u_{cs} \rightarrow V_p$

@ INLET $\boxed{u_r = V_j - V_p}$

@ OUTLET $u_r = 0$ in x-direction

$$-F = -\beta_j \dot{m}_j (V_j - V_p)$$

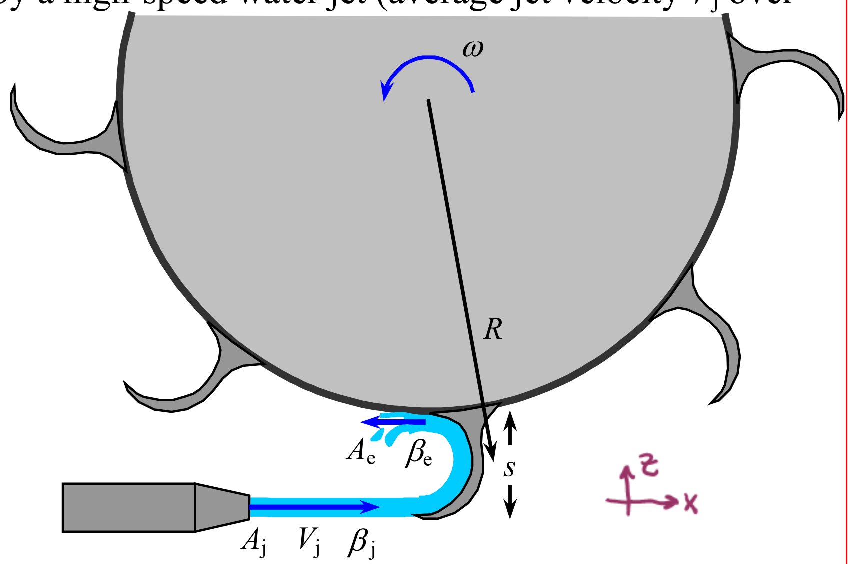
$$\dot{m}_j = \rho_{water} (V_j - V_p) A_j$$

$$\boxed{F = \beta_j \rho_{water} (V_j - V_p)^2 A_j}$$

Answer in variables

Example: Force on a bucket of a Pelton-type (impulse) hydroturbine

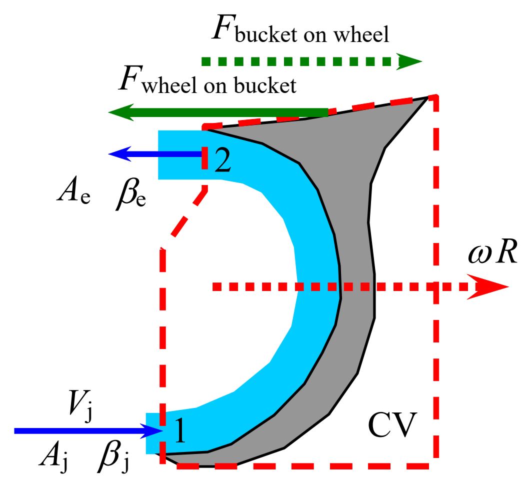
Given: An impulse turbine is driven by a high-speed water jet (average jet velocity V_j over jet area A_j, with momentum flux correction factor β_j) that impinges on turning buckets attached to a turbine wheel as shown. The turbine wheel rotates at angular velocity ω, and is horizontal; therefore, gravity effects are not important in this problem. (The view in the sketch is from the top.) The turning buckets turn the water approximately 180 degrees, and the water exits the bucket over exit cross-sectional area A_e with exit momentum flux correction factor β_e. For simplicity, we approximate that the bucket dimension s is much smaller than turbine wheel radius R ($s \ll R$).



(**a**) **To do**: Calculate the force of the bucket on the turbine wheel, $F_{\text{bucket on wheel}}$, at the instant in time when the bucket is in the position shown.

(**b**) **To do**: Calculate the power delivered to the turbine wheel.

Solution: We choose a control volume surrounding the bucket, cutting through the water jet at the inlet to the bucket, and cutting through the water exiting the bucket. Note that this is a *moving control volume*, moving to the right at speed ωR. We also cut through the welded joint between the bucket and the turbine wheel, where the force $F_{\text{bucket on wheel}}$ is to be calculated. Because of Newton's third law, the force acting *on the control volume* at this location is equal in magnitude, but opposite in direction, and we call it $F_{\text{wheel on bucket}}$.



Since the pressure through an incompressible jet exposed to atmospheric air is equal to P_{atm}, the pressure at the inlet (1) is equal to P_{atm}, and the pressure at the exit (2) is also equal to P_{atm}.

WE APPROXIMATE THIS AS AN INERTIAL REFERENCE FRAME

$\vec{V}_r = \vec{V} - \vec{V}_{cs}$ @ inlets & outlets

At ① (inlet), $\vec{V}_r = V_j \vec{i} - \omega R \vec{i} \rightarrow \boxed{\overbrace{u_{r_1}}^{\text{x-comp}} = V_j - \omega R}$

CONS. OF MASS

- Incompressible
- Quasi-steady   Approximation
- Use relative velocities for the moving CV

$$\sum_{in} \dot{m}_r = \sum_{out} \dot{m}_r$$

$$\cancel{\rho} V_{r,in} A_{in} = \cancel{\rho} V_{r,out} A_{out}$$

$$(V_j - \omega R) A_j = V_{r,out} A_e$$

$$\boxed{V_{r,out} = (V_j - \omega R) \frac{A_j}{A_e}}$$

$$\boxed{u_{r_2} = -(V_j - \omega R) \frac{A_j}{A_e}}$$

X-comp of relative velocity @ outlet (2)

Plug u_{r_1} & u_{r_2} into our mom. eq.

• Use the x-component of the steady linear momentum equation for a moving CV,

$$\sum F_x = \cancel{\sum F_{x,\text{ gravity}}} + \cancel{\sum F_{x,\text{ pressure}}} + \cancel{\sum F_{x,\text{ viscous}}} + \sum F_{x,\text{ other}} = \sum_{\text{out}} \beta \dot{m}_r u_r - \sum_{\text{in}} \beta \dot{m}_r u_r$$

none in x-dir

$P = P_{atm}$ everywhere

wise CV

$\sum F_{x,\text{ other}} = -F_{\text{bucket on wheel}}$

$$\beta_e \dot{m}_r u_{r_2} - \beta_j \dot{m}_r u_{r_1}$$

where $\dot{m}_r = \rho\left(V_j - \omega R\right) A_j$

$$\boxed{F_{\text{bucket on wheel}} = \rho\left(V_j - \omega R\right)^2 A_j\left(\beta_j + \beta_e \frac{A_j}{A_e}\right)}$$

ANS

Force of one bucket acting on the turbine wheel @ the instant in time when jet is aligned with the bucket

APPROXIMATE

(b) Power delivered to the turbine wheel

Soln:

$$\dot{W}_{\text{wheel}} = \text{Torque} \cdot |\text{angular velocity}|$$

$$= F_{\text{bucket on wheel}} \cdot R \cdot \omega$$

$R + \frac{S}{2}$

APPROXIMATE

$$\boxed{\dot{W}_{\text{wheel}} = \rho\left(V_j - \omega R\right)^2 A_j\left(\beta_j + \beta_e \frac{A_j}{A_e}\right) R\omega}$$