STREAM FUNCTION, CYLINDRICAL COORDINATES

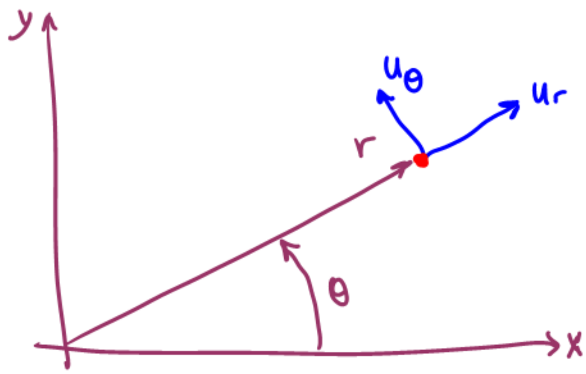
In this lesson, we will:

- Define the **Stream Function in Cylindrical Coordinates**
- Show how to plot **Streamlines** in cylindrical coordinates
- Do some example problems

Definition of Stream Function in Cylindrical Coordinates

TWO POSSIBILITIES:
- Planar Flow (in r-θ plane)
- Axisymmetric Flow (in r-z plane)

- PLANAR FLOW → Same 2-D plane as x-y plane



CONTINUITY: $\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) + \cancel{\frac{\partial u_z}{\partial z}} = 0$

2-D $(u_z = 0)$ $\frac{\partial}{\partial z}(\;) = 0$

mult by r:

$$\boxed{\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_\theta}{\partial \theta} = 0}$$

2-D cont. eq for incomp. flow in r-θ plane

Define ψ: $\boxed{u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}}$; $\boxed{u_\theta = -\frac{\partial \psi}{\partial r}}$ ☆

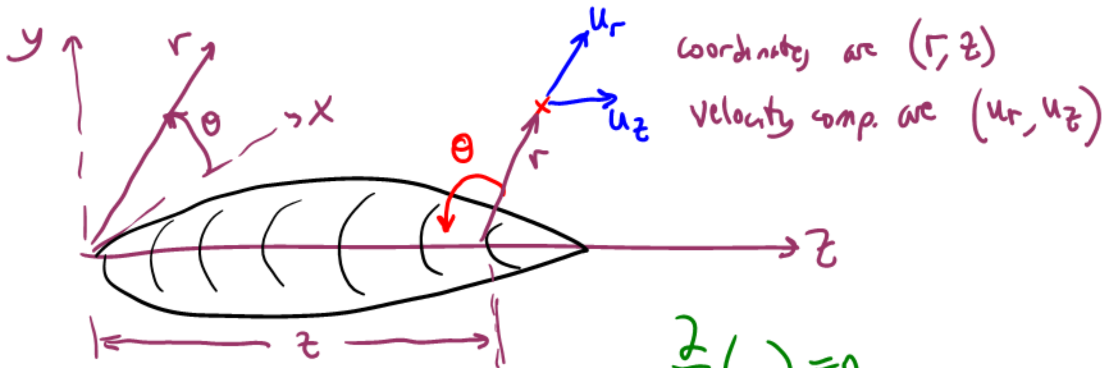
Verify continuity: → $\frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(-\frac{\partial \psi}{\partial r}\right)$

$$\frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{\partial^2 \psi}{\partial \theta \partial r} = 0$$

$\psi = \psi(r, \theta)$

Since order of differentiation doesn't matter cont. satisfied

• AXISYMMETRIC FLOW in r-z plane
with no θ dependence
a "2-D" flow



$$\frac{\partial}{\partial\theta}(\ )=0$$

Continuity:
$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}}^{\,0} + \frac{\partial u_z}{\partial z} = 0$$

Define ψ:
$$\boxed{u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}} \qquad \boxed{u_z = \frac{1}{r}\frac{\partial\psi}{\partial r}}$$

Can verify that any $\psi(r,z)$ satisfies continuity

WE USE PLANAR FLOW IN OUR EXAMPLES

Example: Streamlines in Cylindrical Coordinates and Cartesian Coordinates

Given: A flow field is steady and 2-D in the r-θ plane, and its stream function is given by

$$\boxed{\psi = V_\infty r \sin\theta}$$

To do:

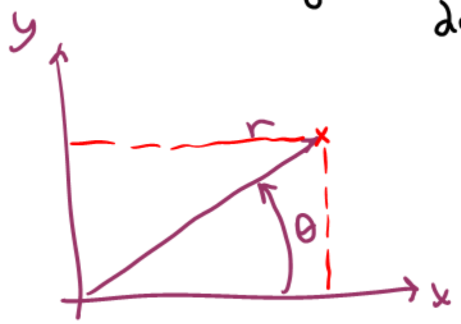
(**a**) Derive expressions for u_r and u_θ and convert to u and v.

(**b**) Sketch some streamlines for this flow field.

Solution:

(a) Planar flow → $u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r} V_\infty r \cos\theta$ $\quad \boxed{u_r = V_\infty \cos\theta}$

$u_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta$ $\quad \boxed{u_\theta = -V_\infty \sin\theta}$

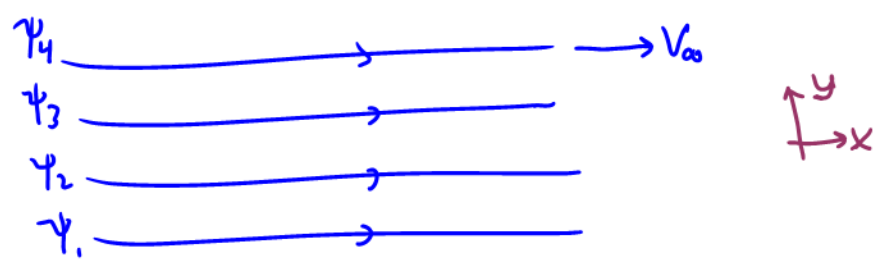


$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$\boxed{\psi(x,y) = V_\infty y}$ ⟶ $u = \frac{\partial\psi}{\partial y} = V_\infty$ $\quad v = -\frac{\partial\psi}{\partial x} = 0$

$$\boxed{u = V_\infty \qquad v = 0}$$

(b) streamlines $\quad$ uniform flow in the x-direction



Example : For $V_\infty = 10.0$ m/s , Calculate $\frac{\dot{V}}{b}$ between streamlines @ $y_1 = 0$ & $y_2 = 5$ m

Sol'n: $\frac{\dot{V}}{b} = \psi_2 - \psi_1 = V_\infty y_2 - V_\infty y_1 = 10.0\,\frac{m}{s}(5-0)\,m = \boxed{50.0\,\frac{m^2}{s}}$

Example: Streamlines in Cylindrical Coordinates

Given: A flow field is steady and 2-D in the r-θ plane, and its velocity field is given by

$$u_r = \frac{c}{r} \qquad u_\theta = 0 \qquad u_z = 0$$

To do: Generate an expression for stream function $\psi(r,\theta)$ and sketch some streamlines.

Solution:

First check continuity

Should be → r not θ (error in the video)

$$\frac{1}{r}\frac{\partial(r u_\theta)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

$$0 + 0 + 0 = 0 \checkmark$$

- Pick one of our ψ definitions

$$u_r = \boxed{\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{c}{r}} \Rightarrow \frac{\partial \psi}{\partial \theta} = c$$

integrate: $\boxed{\psi = c\theta + f(r)}$

- Other ψ definition

$$u_\theta = -\frac{\partial \psi}{\partial r} = -f'(r) = 0$$

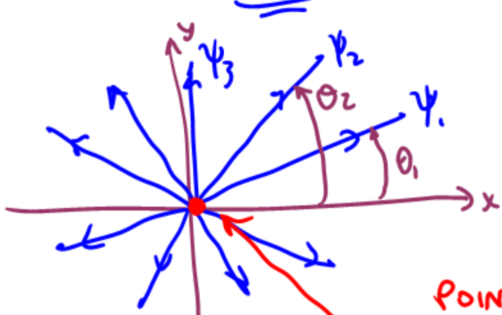
$f(r) = \text{constant}$

$$\therefore \boxed{\psi(r,\theta) = c\theta + \text{constant}}$$

Streamlines: Lines of constant ψ are streamlines

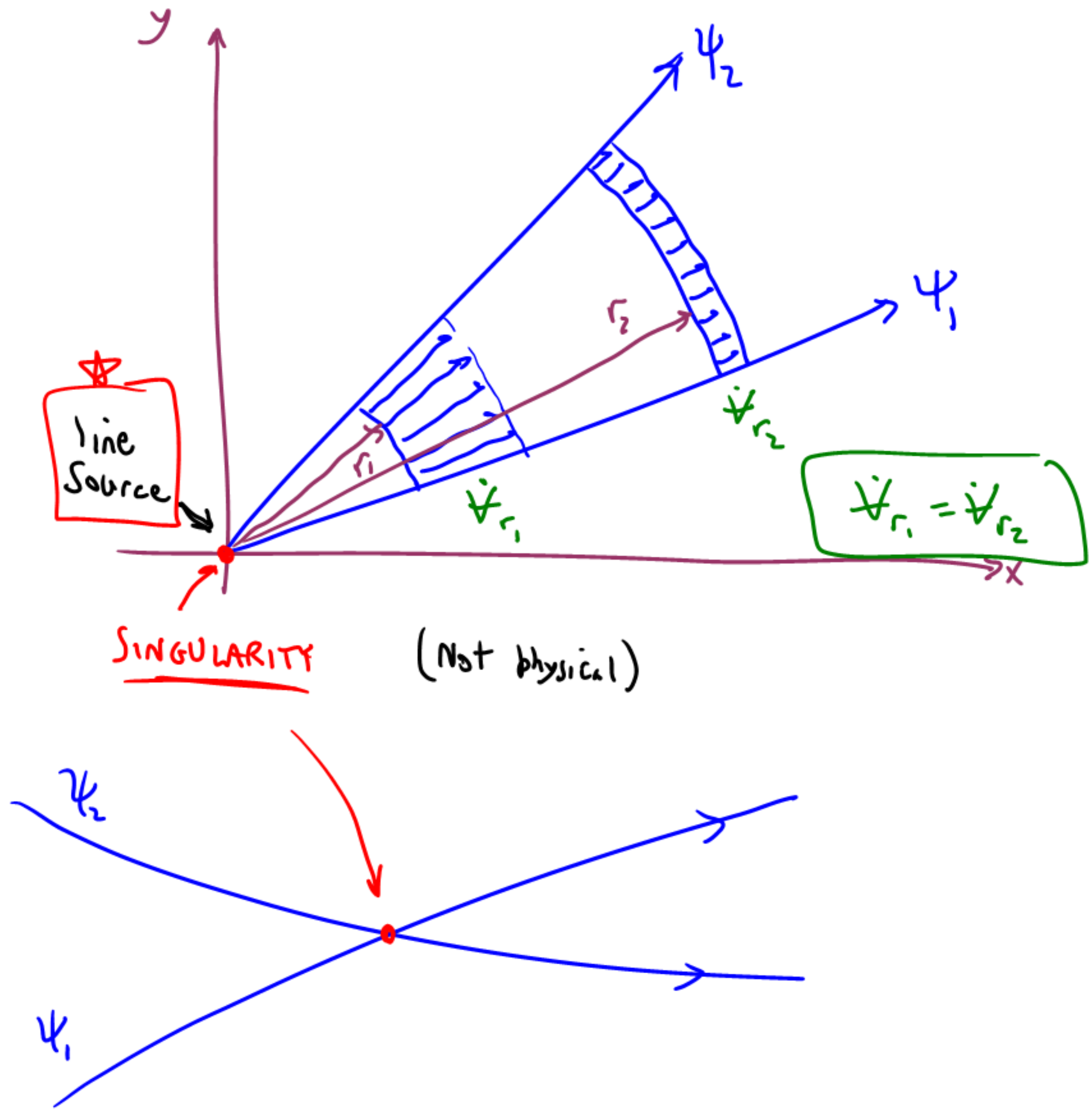
are lines of constant θ

Streamlines here are rays from the origin



★ LINE SOURCE

POINT OF SINGULARITY @ origin



y
ψ_2
ψ_1
r_2
r_1
line Source
$\dot{\psi}_{r_1}$
$\dot{\psi}_{r_2}$
$\dot{\psi}_{r_1} = \dot{\psi}_{r_2}$
x
SINGULARITY
(Not physical)
ψ_2
ψ_1

Example: Velocity Components and Stream Function in Cylindrical Coordinates

Given: A flow field is steady and 2-D in the r-θ plane, and its velocity field is given by

$$u_r = \text{unknown} \qquad u_\theta = \frac{C}{r} \qquad u_z = 0$$

To do: Generate expressions for unknown velocity component u_r and for stream function $\psi(r,\theta)$.

Solution: 2-D continuity $\quad \frac{\partial (r u_r)}{\partial r} + \underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0} = 0$

$$\frac{\partial (r u_r)}{\partial r} = 0 \rightarrow \text{integrate} \quad r u_r = \text{fnc}(\theta)$$

so $\boxed{u_r = \frac{\text{fnc}(\theta)}{r}}$ ANY function of θ will satisfy continuity

Let's pick $\text{fnc}(\theta) = K = \text{constant} \Rightarrow \boxed{u_r = \frac{K}{r}}$ ✱

• Now calculate ψ $\quad \frac{\partial \psi}{\partial r} = -u_\theta = -\frac{C}{r}$

Integrate: $\boxed{\psi = -C \ln r + f(\theta)} \Rightarrow \frac{\partial \psi}{\partial \theta} = f'(\theta)$

Use other def'n: $\frac{\partial \psi}{\partial \theta} = r u_r = r\frac{K}{r} = K$

Equating: $f'(\theta) = K \rightarrow$ integrate: $f(\theta) = K\theta + \text{const}$

$$\boxed{\psi(r,\theta) = -C \ln r + K\theta + \text{const}}$$ ✱

verify

$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{K}{r}$

$u_\theta = -\frac{\partial \psi}{\partial r} = \frac{C}{r}$

You can plot streamlines by choosing ψ values