EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATION

In this lesson, we will:

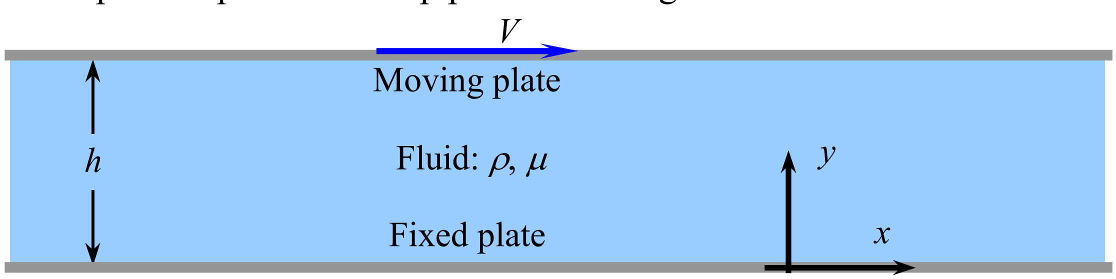
- Review the **Procedure** for solving fluid flow problems using the differential equations of fluid flow (continuity and Navier-stokes)
- Carefully walk though a detailed example problem to illustrate the step-by-step procedure

Review of Procedure for Solving Fluid Flow Problems

Step 1. Identify the flow geometry and flow domain.
Step 2. List assumptions, approximations, and boundary conditions.
Step 3. List all appropriate differential equations and unknowns (and simplify).
Step 4. Solve the equations.
Step 5. Apply the boundary conditions.
Step 6. Verify the results.
Step 7. Calculate other properties of interest.

Example: Exact solution for Couette flow

Given: Steady, incompressible, fully developed laminar flow in the *x*-*y* plane between two infinite parallel plates. The top plate is moving and the bottom one is stationary.



To do: Calculate the velocity field.

Solution:

Step 1. Identify the flow geometry and flow domain.

See sketch ↑ infinite in z-direction
" " x- "

Step 2. List assumptions, approximations, and boundary conditions.

Assumptions and approximations:

1. The flow is steady [$\partial/\partial t$ of anything = 0].
2. The flow is two-dimensional in the *x*-*y* plane [$\partial/\partial z$ of anything = 0, $w = 0$].
3. Gravity effects are negligible or ignored.
4. The flow is fully developed [$\partial/\partial x$ of any velocity = 0; velocity does not change with x].
5. Pressure is constant everywhere.

BC's: • @ $y=0$, $u=v=0$ (no slip)
• @ $y=h$, $u=V$, $v=0$ (")

Step 3. List all appropriate differential equations and unknowns (and simplify).

Continuity:

$$\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{(4)} + \frac{\partial v}{\partial y} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{(2)} = 0 \Rightarrow \boxed{\frac{\partial v}{\partial y} = 0} \qquad (1)$$

x-mom:

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{(1)} + u\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{(4)} + v\frac{\partial u}{\partial y} + w\underbrace{\cancel{\frac{\partial u}{\partial z}}}_{(2)}\right) = -\underbrace{\cancel{\frac{\partial P}{\partial x}}}_{(5)} + \underbrace{\cancel{\rho g_x}}_{(3)}$$

$$+ \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{(4)} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{(2)}\right)$$

$$\boxed{\rho v \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2}} \qquad (2)$$

y-mom:

$$\rho\left(\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{(1)} + u\underbrace{\cancel{\frac{\partial v}{\partial x}}}_{(4)} + v\underbrace{\cancel{\frac{\partial v}{\partial y}}}_{(\text{cont})} + w\underbrace{\cancel{\frac{\partial v}{\partial z}}}_{(2)}\right) = -\underbrace{\cancel{\frac{\partial P}{\partial y}}}_{(5)} + \underbrace{\cancel{\rho g_y}}_{(3)}$$

$$+ \mu\left(\underbrace{\cancel{\frac{\partial^2 v}{\partial x^2}}}_{(4)} + \underbrace{\cancel{\frac{\partial^2 v}{\partial y^2}}}_{(\text{cont})} + \underbrace{\cancel{\frac{\partial^2 v}{\partial z^2}}}_{(2)}\right)$$

$$\boxed{0 = 0} \qquad \text{(Exactly satisfied)}$$

z-mom $\rightarrow$ also gives $\boxed{0=0}$

Step 4. Solve the equations.
Step 5. Apply the boundary conditions.

} DO SIMULTANEOUSLY

Eq.(1) $\rightarrow \frac{\partial v}{\partial y} = 0 \rightarrow \underline{v = f(x)}$

But we know ($\underline{A \& A\ (4)}$) $\frac{\partial}{\partial x}$(velocity comp) $= 0$

i.e, $\underline{v \neq f(x)}$ $\quad \therefore \boxed{v = \text{constant}}$

BC $\rightarrow$ @ $y=0$, $\underline{v=0}$ $\quad$ constant $=0$

$\therefore \boxed{v = 0 \text{ everywhere}}$ ☆

Eq (2) $\rightarrow \rho \cancel{v} \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} \rightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = 0}$ (3)

0 from above

But $u \neq$ fnc(x) $\quad$ A&A (4)
$u \neq$ fnc(t) $\quad$ (1)
$u \neq$ fnc(z) $\quad$ (2)
} $\underline{u = u(y) \text{ only}}$!

(3) becomes $\boxed{\frac{d^2 u}{dy^2} = 0}$ (4)

Integrate (twice) $\rightarrow \frac{du}{dy} = C_1 \rightarrow \boxed{u = C_1 y + C_2}$

Apply BCs : @ $y=0$, $u=0 \rightarrow 0 = 0 + C_2 \Rightarrow \boxed{C_2 = 0}$

@ $y=h$, $u=V \rightarrow V = C_1 h \Rightarrow \boxed{C_1 = \frac{V}{h}}$

Thus, $\boxed{u = V\frac{y}{h}}$ ☆

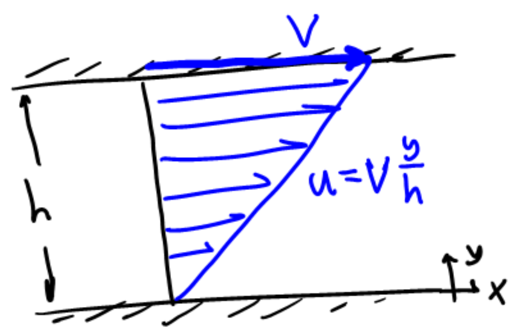
Step 6. Verify the results.

make sure we satisfy all eqs & BCs

BCs: @ $y=0$, $u=0$, $v=0$ ✓

@ $y=h$, $u=V$, $v=0$ ✓

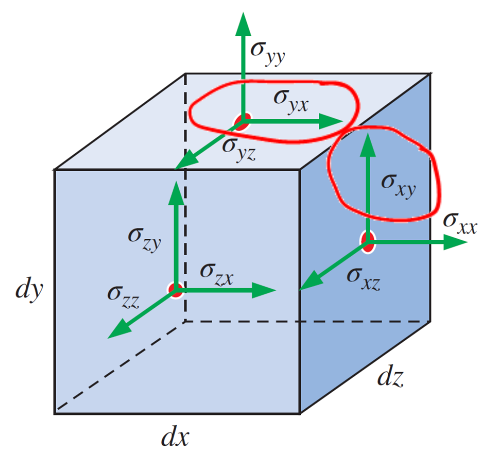
$$\boxed{\vec{V} = V\frac{y}{h}\vec{i} + 0\vec{j}}$$



Couette Flow

Step 7. Calculate other properties of interest.

The stress tensor:

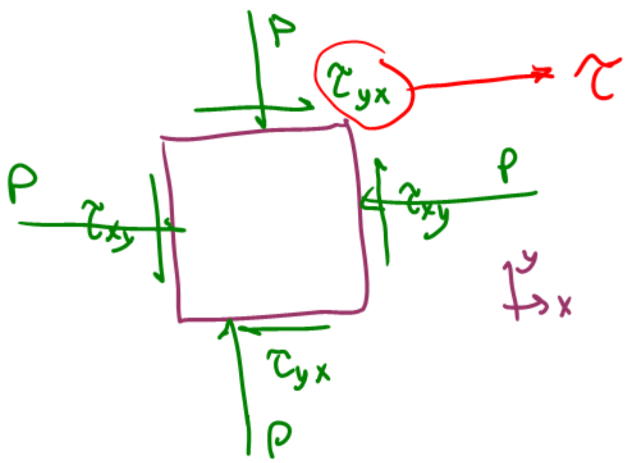


From Çengel and Cimbala, Ed. 4.

$$\sigma_{ij} = \underbrace{\begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}}_{\text{Pressure stress tensor}} + \underbrace{\begin{pmatrix} 2\mu\cancel{\frac{\partial u}{\partial x}} & \mu\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right) & \mu\left(\cancel{\frac{\partial u}{\partial z}} + \cancel{\frac{\partial w}{\partial x}}\right) \\ \mu\left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y}\right) & 2\mu\cancel{\frac{\partial v}{\partial y}} & \mu\left(\cancel{\frac{\partial v}{\partial z}} + \cancel{\frac{\partial w}{\partial y}}\right) \\ \mu\left(\cancel{\frac{\partial w}{\partial x}} + \cancel{\frac{\partial u}{\partial z}}\right) & \mu\left(\cancel{\frac{\partial w}{\partial y}} + \cancel{\frac{\partial v}{\partial z}}\right) & 2\mu\cancel{\frac{\partial w}{\partial z}} \end{pmatrix}}_{\tau_{ij} = \text{viscous stress tensor OR Deviatoric stress tensor}}$$

Simplify for our problem, where $u = V\frac{y}{h}$ $\quad v=0 \quad w=0$

$$\sigma_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} 0 & \mu\frac{du}{dy} & 0 \\ \mu\frac{du}{dy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$\boxed{\tau = \mu\frac{du}{dy}}$ = constant