NAVIER-STOKES SOLUTIONS, CYLINDRICAL COORDINATES

In this lesson, we will:

- Write out the components of the **Continuity** and **Navier-Stokes** equations in **Cartesian Coordinates** and in **Cylindrical Coordinates**
- Discuss an **Alternate Form** of some of the viscous terms in the θ-component of the Navier-Stokes equation
- Do an example problem in cylindrical coordinates – fully developed laminar pipe flow

Continuity and Navier-Stokes Equations, Vector Forms

The **vector** forms of the equations of fluid motion are valid for any coordinate system. Here are the vector equations for the continuity and Navier-Stokes equations:

Continuity Equation: $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0}$ (compressible), $\boxed{\vec{\nabla} \cdot \vec{V} = 0}$ (incompressible)

Incompressible Navier-Stokes Equation: $\boxed{\rho \frac{D\vec{V}}{Dt} = \rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V} \right] = -\vec{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{V}}$

Vorticity: $\boxed{\vec{\zeta} = \vec{\nabla} \times \vec{u}}$

Incompressible Fluid Flow Equations in Cartesian Coordinates (*x*, *y*, *z*), (*u*, *v*, *w*)

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$ Continuity

$$\boxed{\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$ x-comp of N-S

$$\boxed{\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)}$$ y-comp of N-S

$$\boxed{\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\underbrace{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)}_{\nabla^2 w}}$$ z-comp of N-S

Other Useful Equations in Cartesian Coordinates:

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$ Laplacian

$$\boxed{\vec{V} \cdot \vec{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}$$ Term in N-S

$$\boxed{\zeta_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}} \quad \boxed{\zeta_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}} \quad \boxed{\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}$$ Vorticity components

Incompressible Fluid Flow Equations in Cylindrical Coordinates $(r,\ \theta,\ z)$, (u_r, u_θ, u_z)

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) + \frac{\partial}{\partial z}(u_z) = 0}$$ Continuity

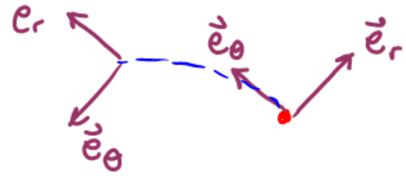
$$\boxed{\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta{}^2}{r}\right) = -\frac{\partial P}{\partial r} + \rho g_r + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)}$$ r-comp

$$\boxed{\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)}$$ θ-comp

$$\boxed{\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\nabla^2 u_z\right)}$$ z-comp of N-S eq.

Other Useful Equations in Cylindrical Coordinates:

$$\boxed{\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}}$$ Laplacian



$$\boxed{\vec{V}\cdot\vec{\nabla} = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}}$$ Term in N-S

$$\boxed{\zeta_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}} \quad \boxed{\zeta_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}} \quad \boxed{\zeta_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}}$$ components of vorticity vector

Alternate Form of the Viscous Term in the θ-Component of the N-S Equation

Apply the above Laplacian operator to the θ-component of the Navier-Stokes equation:

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)$$

- use product rule
- use inverse product rule

These two terms can be written as

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)$$

$$\boxed{\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)}$$

Example: Fully Developed Laminar Flow in a Round Pipe

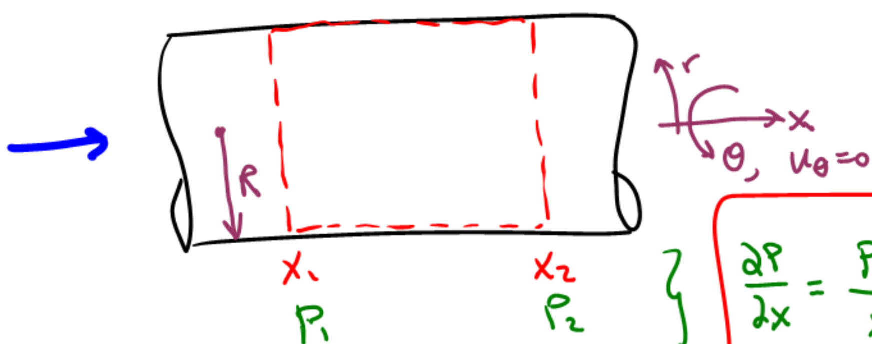
Given: A fluid of density ρ and viscosity μ flows through a smooth round pipe of inner radius R. The pipe is long and we are interested in the fully developed region of the flow. In this region, the pressure decreases *linearly* with distance x downstream.

To do: Generate an expression for the velocity profile.

Solution: By convention, we use x and u instead of z and u_z for the flow direction. We apply our step-by-step procedure:

Step 1. Identify the flow geometry and flow domain.
Step 2. List assumptions, approximations, and boundary conditions.
Step 3. List all appropriate differential equations and unknowns (and simplify).
Step 4. Solve the equations.
Step 5. Apply the boundary conditions.
Step 6. Verify the results.
Step 7. Calculate other properties of interest.

Step 1:



$$\boxed{\frac{\partial P}{\partial x} = \frac{P_2 - P_1}{x_2 - x_1} = \text{constant}}$$

Step 2: A & A:

1) Pipe is very long in x-direction
2) Steady
→ 3) $u_r = 0$
4) Incompressible, Newtonian w/ constant properties (ρ, μ) & LAMINAR ☆
5) Constant pressure gradient in x-direction
6) Flow is axisymmetric with no swirl
$\frac{\partial}{\partial \theta} = 0$, $u_\theta = 0$
7) Ignore gravity

Step 3:

Continuity:
$$\frac{1}{r}\cancel{\frac{\partial}{\partial r}(r u_r)} + \frac{1}{r}\cancel{\frac{\partial}{\partial \theta}(u_\theta)} + \frac{\partial}{\partial x}(u) = 0 \quad \Rightarrow \quad \boxed{\frac{\partial u}{\partial x} = 0}$$

(3) (6)

r-mom
$$\rho\left(\cancel{\frac{\partial u_r}{\partial t}} + u_r \cancel{\frac{\partial u_r}{\partial r}} + \frac{u_\theta}{r}\cancel{\frac{\partial u_r}{\partial \theta}} + u\cancel{\frac{\partial u_r}{\partial x}} - \cancel{\frac{u_\theta^2}{r}}\right) = -\frac{\partial P}{\partial r} + \cancel{\rho g_r} + \mu\left(\cancel{\nabla^2 u_r} - \cancel{\frac{u_r}{r^2}} - \frac{2}{r^2}\cancel{\frac{\partial u_\theta}{\partial \theta}}\right)$$

$\frac{\partial P}{\partial r} = 0$

$u_r = 0$ —— assume (3) —— (6) (7) (3) (3) (6)

θ-mom
$$\rho\left(\cancel{\frac{\partial u_\theta}{\partial t}} + u_r \cancel{\frac{\partial u_\theta}{\partial r}} + \frac{u_\theta}{r}\cancel{\frac{\partial u_\theta}{\partial \theta}} + u\cancel{\frac{\partial u_\theta}{\partial x}} + \cancel{\frac{u_r u_\theta}{r}}\right) = -\frac{1}{r}\cancel{\frac{\partial P}{\partial \theta}} + \cancel{\rho g_\theta} + \mu\left(\cancel{\nabla^2 u_\theta} - \cancel{\frac{u_\theta}{r^2}} + \frac{2}{r^2}\cancel{\frac{\partial u_r}{\partial \theta}}\right)$$

(6) (6) (7) (6) (6) (6)

X-mom
$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + u_r \cancel{\frac{\partial u}{\partial r}} + \frac{u_\theta}{r}\cancel{\frac{\partial u}{\partial \theta}} + u\cancel{\frac{\partial u}{\partial x}}\right) = -\frac{\partial P}{\partial x} + \cancel{\rho g_x} + \mu\left(\nabla^2 u\right)$$

(2) (3) (6) (cont) (7)

$$\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 u}{\partial \theta^2}} + \cancel{\frac{\partial^2 u}{\partial x^2}}\right)$$

(7) (cont)

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x}}$$

P not function of t, θ, r
$P = P(x)$ only

$u \neq$ fnc of t
$u \neq$ fnc of θ
$u \neq$ fnc of x
$\Big\}$ $u = u(r)$ only

$$\boxed{\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{dP}{dx}}$$

Known constant

B.C.s $\boxed{@\ r = R,\ u = 0}$

$@\ r = 0,\ u = \max \Rightarrow \boxed{\frac{du}{dr} = 0 \ @\ r = 0}$

Need 1 BC for P @ some x location

Step 4

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{dP}{dx}$$

$$\times r \rightarrow \quad \frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{r}{\mu}\frac{dP}{dx}$$

Int: $$\boxed{r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1}$$

$$\div r \rightarrow \quad \frac{du}{dr} = \frac{r}{2\mu}\frac{dP}{dx} + \frac{C_1}{r}$$

Int: $$\boxed{u = \frac{r^2}{4\mu}\frac{dP}{dx} + C_1 \ln r + C_2}$$

Step 5:

@ $r=0$ $\quad \frac{du}{dr} = 0$

$$0 = 0 + C_1 \rightarrow \boxed{C_1 = 0}$$

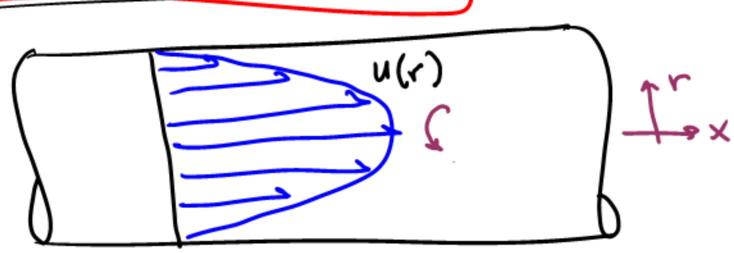
@ $r=R$, $u=0$

$$0 = \frac{R^2}{4\mu}\frac{dP}{dx} + C_2 \rightarrow \boxed{C_2 = -\frac{R^2}{4\mu}\frac{dP}{dx}}$$

$$u = \frac{r^2}{4\mu}\frac{dP}{dx} - \frac{R^2}{4\mu}\frac{dP}{dx}$$

$$☆ \quad \boxed{u = \frac{1}{4\mu}\frac{dP}{dx}\left(r^2 - R^2\right)} \quad ☆$$

A paraboloid



Step 6: Verify → all eqs & BCs are satisfied ✓

Step 7 Other properties of interest

- $u_{max} = u$ @ $r=0$

$$\boxed{u_{max} = -\frac{R^2}{4\mu}\frac{dP}{dx}}$$

- $$\dot{\forall} = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} u \, r\,dr\,d\theta \Rightarrow$$

$$\boxed{\dot{\forall} = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}}$$

- $V = \frac{\dot{\forall}}{A}$ = average speed $= \frac{\dot{\forall}}{\pi R^2}$

$$\boxed{V = -\frac{R^2}{8\mu}\frac{dP}{dx}}$$

- τ = shear stress $= \tau_{rx} = \tau_{xr}$

$$\tau_{ij} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rx} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta x} \\ \tau_{xr} & \tau_{x\theta} & \tau_{xx} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mu\frac{du}{dr} \\ 0 & 0 & 0 \\ \mu\frac{du}{dr} & 0 & 0 \end{pmatrix}$$

$$\boxed{\tau = \tau_{rx} = \mu\frac{du}{dr} = \frac{R}{2}\frac{dP}{dx}}$$

At the wall

Can calc f

$$\boxed{f = \frac{64}{Re}}$$