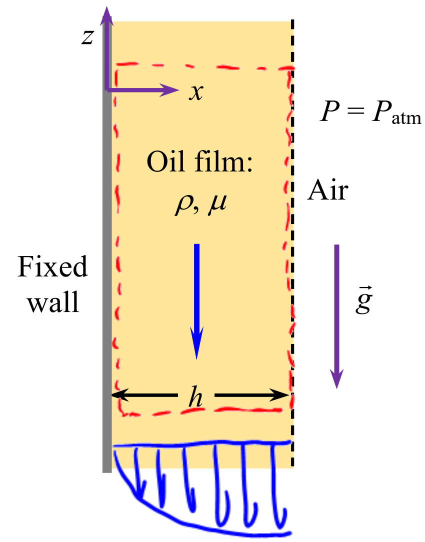
MORE SOLUTIONS OF THE NAVIER-STOKES EQUATION

In this lesson, we will:

- Do two more example problems that are exact solutions of the Navier-Stokes equation

Example in Cartesian Coordinates: Oil Film on a Vertical Wall



Given: Consider steady, incompressible, parallel, laminar flow of a film of oil falling slowly down an infinite vertical wall as sketched. The oil film thickness is h, and gravity acts in the negative z direction (downward in the figure). There is no applied (forced) pressure driving the flow – the oil falls by gravity alone.

To do: Calculate the velocity and pressure fields in the oil film and sketch the normalized velocity profile. You may neglect changes in the hydrostatic pressure of the surrounding air.

Solution: We apply our step-by-step procedure:

Step 1. Identify the flow geometry and flow domain.

see diagram

Step 2. List assumptions, approximations, and boundary conditions.

Assumptions and Approximations:

1. The wall is infinite in the y-z plane (y is into the page for a right-handed coordinate system). Flow is fully developed
2. The flow is steady, so all partial derivatives with respect to time are zero. $\frac{\partial}{\partial t} = 0$
3. The flow is parallel (the x component of velocity, u, is zero everywhere). $u = 0$
4. The fluid is incompressible and Newtonian, and the flow is laminar.
5. Pressure $P = P_{atm}$ = constant at the free surface. In other words, there is no applied pressure gradient pushing the flow; the flow establishes itself due to a balance between gravitational forces and viscous forces. In addition, since there is no gravity force in the horizontal direction, *$P = P_{atm}$ everywhere.*
6. The velocity field is purely two-dimensional, which implies that $v = 0$ and all partial derivatives with respect to y are zero. $\frac{\partial}{\partial y} = 0$
7. The components of gravity are $g_x = g_y = 0$ and $g_z = -g$.

Boundary Conditions:

@ wall, no slip $\Rightarrow$ $u = v = w = 0$ @ $x = 0$

@ free surface $\Rightarrow$ $\frac{\partial w}{\partial x} = 0$ @ $x = h$

interface w/ $\mu_{oil} \gg \mu_{air}$

Step 3. List all appropriate differential equations and unknowns (and simplify).

$$\cancel{\frac{\partial u}{\partial x}}+\cancel{\frac{\partial v}{\partial y}}+\frac{\partial w}{\partial z}=0 \qquad \text{continuity} \quad \Rightarrow \boxed{\frac{\partial w}{\partial z}=0} \quad (1)$$

(3) (6)

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}+\cancel{u\frac{\partial u}{\partial x}}+\cancel{v\frac{\partial u}{\partial y}}+\cancel{w\frac{\partial u}{\partial z}}\right)=\cancel{-\frac{\partial P}{\partial x}}+\cancel{\rho g_x}+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\cancel{\frac{\partial^2 u}{\partial y^2}}+\cancel{\frac{\partial^2 u}{\partial z^2}}\right) \qquad \text{x-mom} \ ✓ \quad 0=0$$

(3) (3) (3) (3) (5) (7) (3) (3) (3)

$$\rho\left(\cancel{\frac{\partial v}{\partial t}}+\cancel{u\frac{\partial v}{\partial x}}+\cancel{v\frac{\partial v}{\partial y}}+\cancel{w\frac{\partial v}{\partial z}}\right)=\cancel{-\frac{\partial P}{\partial y}}+\cancel{\rho g_y}+\mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}+\cancel{\frac{\partial^2 v}{\partial y^2}}+\cancel{\frac{\partial^2 v}{\partial z^2}}\right) \qquad \text{y-mom} \quad 0=0 \ ✓$$

(6) (6) (6) (6) (5) (7) (6) (6) (6)

$$\rho\left(\cancel{\frac{\partial w}{\partial t}}+\cancel{u\frac{\partial w}{\partial x}}+\cancel{v\frac{\partial w}{\partial y}}+\cancel{w\frac{\partial w}{\partial z}}\right)=\cancel{-\frac{\partial P}{\partial z}}+\underbrace{\rho g_z}_{-\rho g}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\cancel{\frac{\partial^2 w}{\partial y^2}}+\cancel{\frac{\partial^2 w}{\partial z^2}}\right) \qquad \text{z-mom}$$

(2) (3) (6) Eq.(1) (5) (6) Eq.(1)

W not fnc of t
W not fnc of y
W not fnc of z

$$\frac{\partial^2 w}{\partial x^2}=\frac{\rho g}{\mu} \Rightarrow \boxed{\frac{d^2 w}{dx^2}=\frac{\rho g}{\mu}}$$

Step 4. Solve the equations.

$$\frac{dw}{dx}=\frac{\rho g}{\mu}x+C_1$$

$$\boxed{W=\frac{\rho g}{2\mu}x^2+C_1 x+C_2}$$

Step 5. Apply the boundary conditions.

$$@\ x=0,\ W=0 \rightarrow 0=0+0+C_2 \rightarrow \boxed{C_2=0}$$

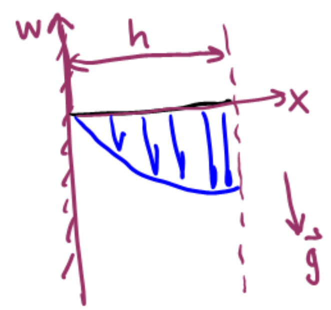
$$@\ x=h,\ \frac{dw}{dx}=0$$

$$0=\frac{\rho g}{\mu}h+C_1 \rightarrow \boxed{C_1=-\frac{\rho g h}{\mu}} \Rightarrow \boxed{W=\frac{\rho g x}{2\mu}(x-2h)} \ *$$

Step 6. Verify the results. ✓

nondimensionally, $x^*=\frac{x}{h}$

$w^*=\frac{w\mu}{\rho g h^2}$



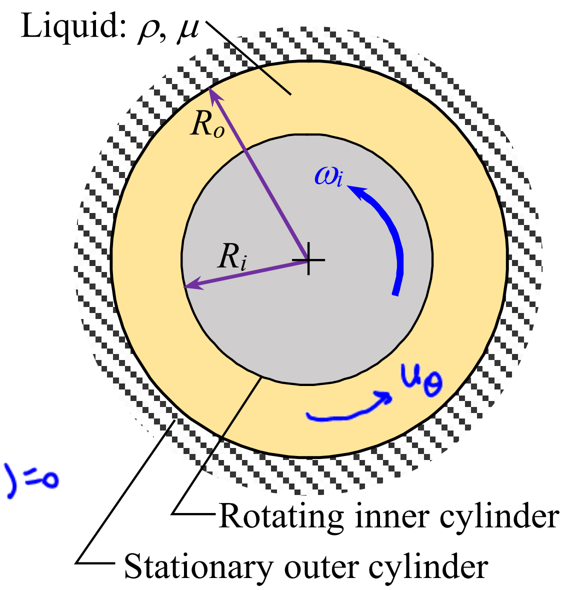
Step 7. Calculate other properties of interest.

Can calc. $\tau_w=\mu\left.\frac{dw}{dx}\right)_w$, Volume flow rate, etc.

Example in Cylindrical Coordinates: Flow Between Two Concentric Cylinders

Given: An incompressible Newtonian liquid is confined between two concentric circular cylinders of infinite length (into the page; the z axis is out of the page) – a solid inner cylinder of radius R_i and a hollow, stationary outer cylinder of radius R_o. The inner cylinder rotates at angular velocity ω_i. The flow is steady, laminar, and two-dimensional in the r-θ plane. The flow is also *rotationally symmetric*, meaning that nothing is a function of coordinate θ (u_θ and P are functions of radius r only). The flow is also circular, meaning that velocity component $u_r = 0$ everywhere.

$\frac{\partial}{\partial \theta}(\;) = 0$



To do: Generate an exact expression for velocity component u_θ as a function of radius r and the other parameters in the problem. You may ignore gravity. $\rightarrow u_\theta(r)$

Solution: We apply our step-by-step procedure:

Step 1. Identify the flow geometry and flow domain. $\rightarrow$ yellow region

Step 2. List assumptions, approximations, and boundary conditions.

Assumptions and Approximations:

1 The cylinders are infinite in the z direction (z is out of the page in the figure of the problem statement for a right-handed coordinate system). The velocity field is purely two-dimensional, which implies that $w = 0$ and derivatives of any velocity component with respect to z are zero. $\frac{\partial}{\partial z}(\;) = 0$

2 The flow is steady, meaning that all time derivatives are zero. $\frac{\partial}{\partial t} = 0$

3 The flow is circular, meaning that the radial velocity component u_r is zero. $u_r = 0$

4 The flow is rotationally symmetric, meaning that nothing is a function of θ. $\frac{\partial}{\partial \theta} = 0$

5 The fluid is incompressible and Newtonian, and the flow is laminar.

6 Gravitational effects are ignored. (Note that gravity may act in the z direction, leading to an additional hydrostatic pressure distribution in the z direction. This would not affect the present analysis.) $\vec{g} = 0$

Boundary Conditions:

No slip $\rightarrow$ @ $r = R_i$, $u_\theta = \omega_i R_i$ (rotating inner cylinder)

@ $r = R_o$, $u_\theta = 0$ (non-rotating outer cylinder)

Step 3. List all appropriate differential equations and unknowns (and simplify).

$$\underbrace{\cancel{\frac{1}{r}\frac{\partial}{\partial r}(r u_r)}}_{(3)} + \underbrace{\cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta)}}_{(4)} + \underbrace{\cancel{\frac{\partial}{\partial z}(u_z)}}_{(1)} = 0$$

continuity $\Rightarrow \boxed{0=0}$ ✓

$$\rho\left(\underbrace{\cancel{\frac{\partial u_r}{\partial t}}}_{(3)} + \underbrace{\cancel{u_r\frac{\partial u_r}{\partial r}}}_{(3)} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}}_{(3)} + \underbrace{\cancel{u_z\frac{\partial u_r}{\partial z}}}_{(3)} - \frac{u_\theta^2}{r}\right)$$

$$= -\frac{\partial P}{\partial r} + \underbrace{\cancel{\rho g_r}}_{(6)} + \mu\left(\underbrace{\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)}}_{(3)} + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}}}_{(3)} + \underbrace{\cancel{\frac{\partial^2 u_r}{\partial z^2}}}_{(3)} - \underbrace{\cancel{\frac{u_r}{r^2}}}_{(3)} - \underbrace{\cancel{\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}}}_{(4)}\right)$$

$$\boxed{\frac{\partial P}{\partial r} = \rho\frac{u_\theta^2}{r}} \rightarrow \boxed{\frac{dP}{dr} = \rho\frac{u_\theta^2}{r}} \ *$$

r-mom

$P \neq$ fnc (t, θ, z)

$P = P(r)$ only

$W = u_z$

$$\rho\left(\underbrace{\cancel{\frac{\partial u_z}{\partial t}}}_{(1)} + \underbrace{\cancel{u_r\frac{\partial u_z}{\partial r}}}_{(1)} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}}_{(1)} + \underbrace{\cancel{u_z\frac{\partial u_z}{\partial z}}}_{(1)}\right) = -\underbrace{\cancel{\frac{\partial P}{\partial z}}}_{\text{ignore}} + \underbrace{\cancel{\rho g_z}}_{(6)} + \mu\left(\underbrace{\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)}}_{(1)} + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}}}_{(1)} + \underbrace{\cancel{\frac{\partial^2 u_z}{\partial z^2}}}_{(1)}\right)$$

z-mom

$$\rho\left(\underbrace{\cancel{\frac{\partial u_\theta}{\partial t}}}_{(2)} + \underbrace{\cancel{u_r\frac{\partial u_\theta}{\partial r}}}_{(3)} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}}}_{(4)} + \underbrace{\cancel{u_z\frac{\partial u_\theta}{\partial z}}}_{(1)} + \underbrace{\cancel{\frac{u_r u_\theta}{r}}}_{(3)}\right)$$

$$= -\underbrace{\cancel{\frac{1}{r}\frac{\partial P}{\partial \theta}}}_{(4)} + \underbrace{\cancel{\rho g_\theta}}_{(6)} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}}}_{(4)} + \underbrace{\cancel{\frac{\partial^2 u_\theta}{\partial z^2}}}_{(1)} - \frac{u_\theta}{r^2} - \underbrace{\cancel{\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}}}_{(3)}\right)$$

θ-mom

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) = 0 \Rightarrow$$

$$\boxed{\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r u_\theta)\right) = 0}$$

Step 4. Solve the equations.

Int. $\quad \frac{1}{r}\frac{d}{dr}(r u_\theta) = C_1$

$\times r \quad \frac{d}{dr}(r u_\theta) = C_1 r$

Int $\quad r u_\theta = \frac{C_1 r^2}{2} + C_2$

$\div r \quad \boxed{u_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}} \ *$

Step 5. Apply the boundary conditions.

$$@\ \underline{r=R_o,\ u_\theta=0} \Rightarrow 0=\frac{C_1R_o}{2}+\frac{C_2}{R_o} \Rightarrow \boxed{C_2=-C_1\frac{R_o^2}{2}}$$

$$@\ \underline{r=R_i,\ u_\theta=\omega R_i} \Rightarrow \omega R_i=\frac{C_1R_i}{2}+\frac{C_2}{R_i}$$

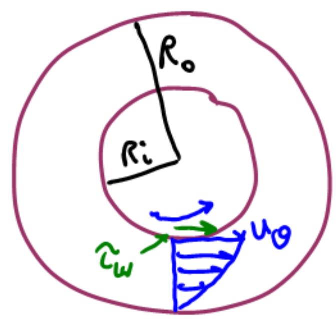
$$\omega R_i=\frac{C_1R_i}{2}-C_1\frac{R_o^2}{2R_i}$$

Solve for C_1

$$\boxed{C_1=\frac{-2R_i^2\omega_i}{R_o^2-R_i^2}} \qquad \boxed{C_2=\frac{R_o^2R_i^2\omega_i}{R_o^2-R_i^2}}$$

Thus

$$\boxed{u_\theta=\frac{R_i^2\omega_i}{R_o^2-R_i^2}\left(\frac{R_o^2}{r}-r\right)} \ \star$$



Step 6. Verify the results. ✓

Step 7. Calculate other properties of interest.

Can calculate
- shear stress @ the walls
- volume flow rate @ a cross-section
- Torque required to rotate the inner cylinder

As gap gets very small, $R_o-R_i \ll R_i$