APPROXIMATION FOR INVISCID REGIONS OF FLOW

In this lesson, we will:

- Define an **Inviscid Region of Flow** and the **Euler Equation**
- Emphasize the difference between **Inviscid Fluid** and **Inviscid Flow**
- Derive the **Beloved Bernoulli Equation** from the Euler equation
- Do an example problem using both Euler and Bernoulli equations

Inviscid Region of Flow

Definition: An inviscid region of flow is a region of flow in which net viscous forces are negligible compared to pressure and/or inertial forces.

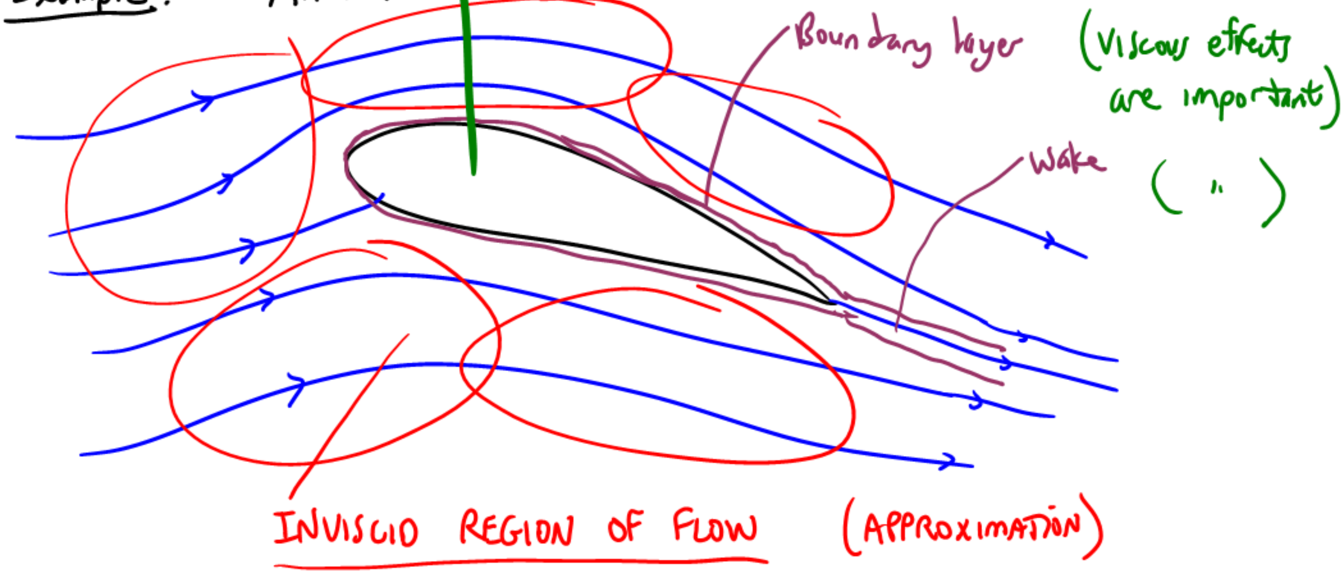
"Inviscid" does not mean that the fluid itself has zero viscosity

↳ However, in some regions of a flow, net viscous effects can be negligible compared to inertial and/or pressure effects

THIS IS WHAT WE CALL AN INVISCID REGION OF FLOW

★ $\boxed{\text{WE MAY HAVE INVISCID FLOWS BUT NOT INVISCID FLUIDS}}$

Example: Airfoil



Approximate Navier-Stokes Equation for an Inviscid Region of Flow

N-S eq in nondimensional form:

$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^* = -[\mathrm{Eu}]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \cancel{\left[\frac{1}{\mathrm{Re}}\right]\vec{\nabla}^{*2}\vec{V}^*}$$

Unsteady Inertial Pressure Gravitational ~~Viscous~~

Recall for creeping flow approximation, $Re \ll 1$

only pressure & viscous terms remain

For inviscid regions of flow, $Re \gg 1$

N-S eq reduces to

$$\boxed{\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g}}$$

☆ EULER EQ (N-S w/o the viscous term)

☆ THIS IS THE APPROXIMATE EQ. WE SOLVE IN INVISCID REGIONS OF FLOW

Derivation of the Beloved Bernoulli Equation from the Euler Equation

We start with the Euler equation for steady incompressible flow,

$$\rho\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = -\vec{\nabla}P + \rho\vec{g}$$

$\vec{\zeta}$ = vorticity vector

VECTOR IDENTITY:

$$\boxed{(\vec{V}\cdot\vec{\nabla})\vec{V} = \vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V}\times(\vec{\nabla}\times\vec{V})}$$

re-write:

$$\vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\zeta} = -\frac{\vec{\nabla}P}{\rho} + \vec{g}$$

$= -\vec{\nabla}\left(\frac{P}{\rho}\right)$

z, $\vec{g}$

$\vec{\nabla}z = \vec{k}$

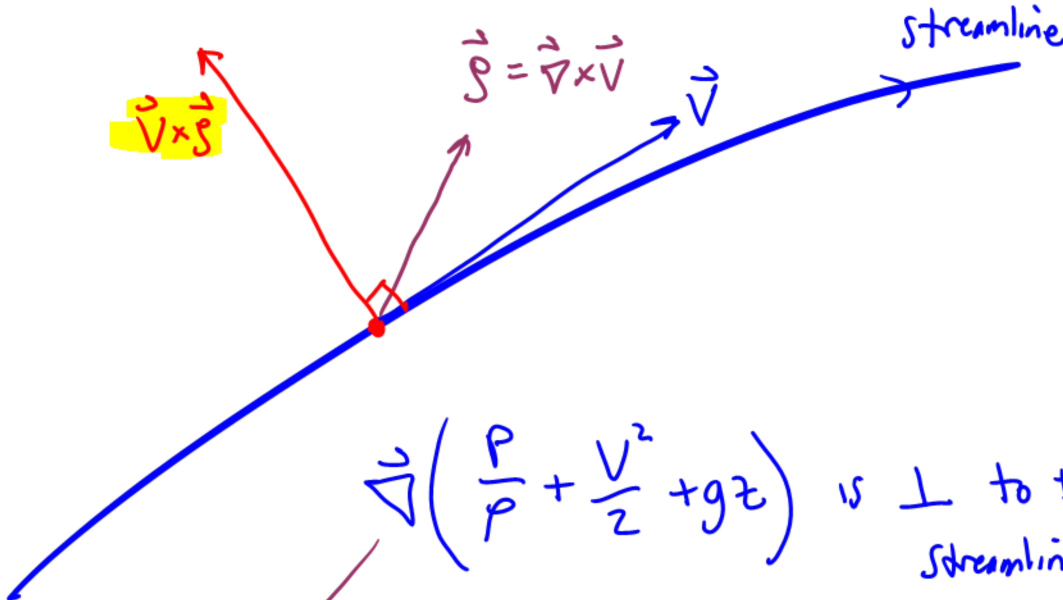
$\vec{g} = -g\vec{k}$

So, $\vec{g} = -g\vec{\nabla} z = -\vec{\nabla}(gz)$

Thus, $\vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\zeta} = -\vec{\nabla}\left(\frac{p}{\rho}\right) - \vec{\nabla}(gz)$

$$\boxed{\vec{\nabla}\left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right) = \vec{V}\times\vec{\zeta}}$$

ALTERNATE FORM OF EULER EQ



$\vec{\nabla}\left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)$ is $\perp$ to the streamline

Gradient operator $\vec{\nabla}$:

$\vec{\nabla}(B)$ is in the direction of maximum change of B

$\vec{\nabla}(B) = 0$ along curves (or surfaces) of constant B

Here, let $B = \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)$

BELOVED BERNOLLI EQ

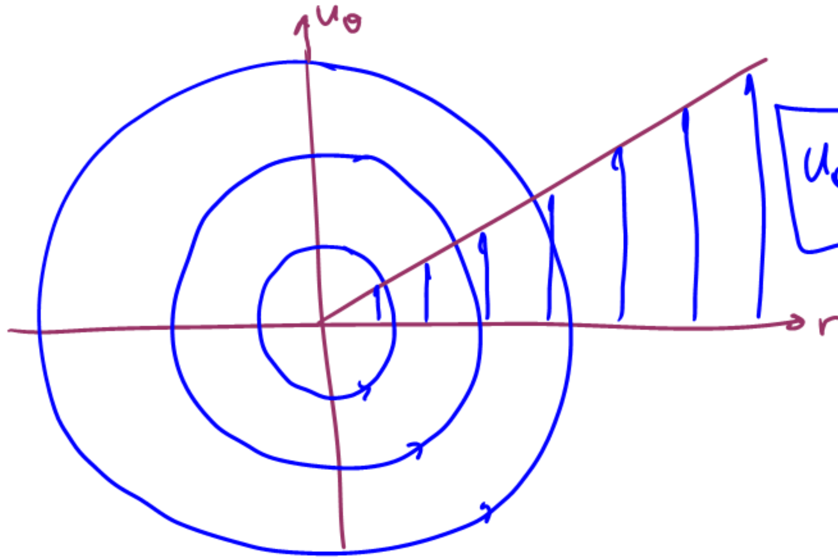
WE CONCLUDE: $\boxed{\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline}}$

Example: Pressure Field in Solid Body Rotation

Given: A fluid is rotating as a solid body (solid body rotation) with angular speed ω perpendicular to the r-θ plane. The flow is steady and gravitational effects are ignored.

To do: Generate an expression for the pressure field.

Solution:



$$\boxed{u_\theta = \omega r}$$

$u_r = 0$

$u_z = 0$

Since the fluid moves as a solid body, there are no viscous forces (fluid particles do not rub against each other)

$\boxed{\text{THIS IS TRULY AN INVISCID FLOW !}}$

THE EULER EQ. APPLIES EVERYWHERE

A & A
(1) steady
(2) ignore gravity
(3) $u_\theta = u_\theta(r)$ only & $\boxed{P = P(r)}$ only

$$\boxed{u_\theta = \omega r}$$

Euler:

$$\underbrace{\cancel{\rho \frac{\partial \vec{V}}{\partial t}}}_{(1)} + \boxed{\rho(\vec{V}\cdot\vec{\nabla})\vec{V} = -\vec{\nabla}P} + \underbrace{\cancel{\rho \vec{g}}}_{(2)}$$

Use the r-component of Euler to solve for P

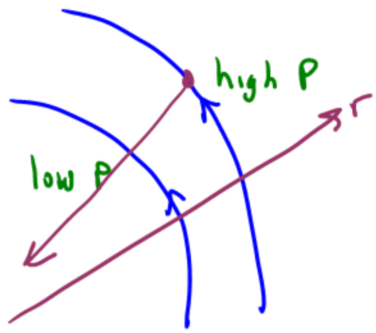
IN $(r$-$\theta)$ coordinates, r-component

$$\boxed{\frac{\partial P}{\partial r} = \rho \frac{u_\theta^2}{r}} \qquad u_\theta = \omega r$$

$P = P(r)$ only $\Rightarrow$ $\frac{dP}{dr} = \rho \frac{\omega^2 r^2}{r}$ $\boxed{\frac{dP}{dr} = \rho \omega^2 r}$

- integrate w.r.t. r $\Rightarrow$ $P = \rho \frac{\omega^2 r^2}{2} + C_1$

$\underline{BC}$: Let $P = P_0$ @ the origin $(r=0)$ $\boxed{C_1 = P_0}$



$$\boxed{P = P_0 + \rho \frac{\omega^2 r^2}{2}}$$ ☆ ANSWER

BELOVED BERNOULLI EQ

$$\frac{P}{\rho} + \frac{V^2}{2} + \cancel{gz} = \text{constant along streamlines}$$

(2)

$\omega^2 r^2$

$\times \rho$

$$P + \frac{\rho \omega^2 r^2}{2} = \text{constant along a streamline}$$

Compare to our solution for P, $\boxed{P = P_0 + \frac{\rho\omega^2 r^2}{2}}$ agrees ☺

$$\boxed{P = -\frac{\rho\omega^2 r^2}{2} + \text{constant along a streamline}}$$

equate these two equations,

$$\boxed{\text{constant along a streamline (Bernoulli constant)} = P_0 + \rho\omega^2 r^2}$$

Our final Bernoulli eq. is $\boxed{P = -\frac{\rho\omega^2 r^2}{2} + P_0 + \rho\omega^2 r^2} = P_0 + \frac{\rho\omega^2 r^2}{2}$ ☆