FANNO FLOW – COMPRESSIBLE DUCT FLOW WITH FRICTION

In this lesson, we will:

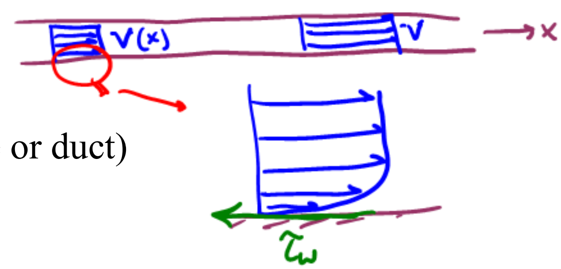
- Introduce **Fanno flow**: flow in a duct with ***friction*** but no heat transfer
- Discuss Fanno flow qualitatively and quantitatively
- Do an example problem

Disclaimer: This is an ***abbreviated*** summary of Fanno flow; a more rigorous analysis is presented in my compressible flow course (ME 420 at Penn State University)

Fanno Flow Introduction, Approximations, and Assumptions

- Steady flow in a pipe or duct
- One-D flow (V approximately constant at any cross-section of the duct, i.e., at any x location; so, $V = V(x)$ only)
- Ideal gas
- Constant gas properties (k, c_P, R, etc.)
- Constant area (long, straight section of pipe or duct)
- Fully developed (ignore entrance effects)
- Negligible heat transfer to or from the gas

ADIABATIC



Comparison with Rayleigh Flow

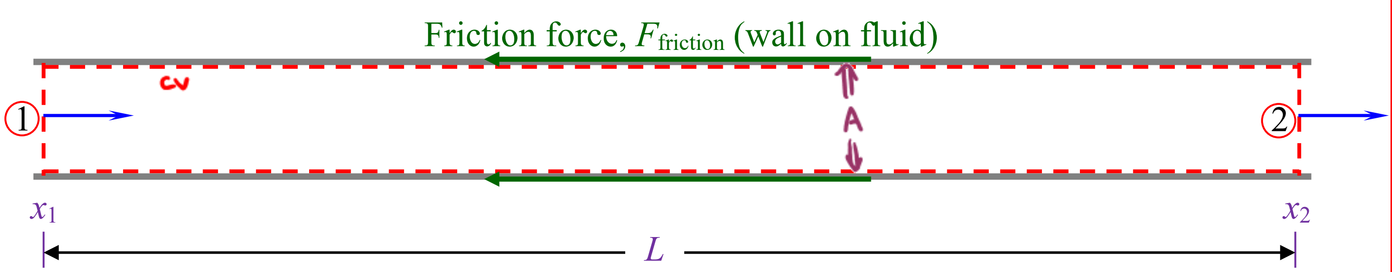
Rayleigh flow

- SHORT DUCTS
- NEGLECT FRICTION
- HEAT TRANSFER IS IMPORTANT

Fanno flow

- LONG DUCTS
- FRICTION IS IMPORTANT
- NEGLECT HEAT TRANSFER

Control Volume Analysis and the Fanno Curve

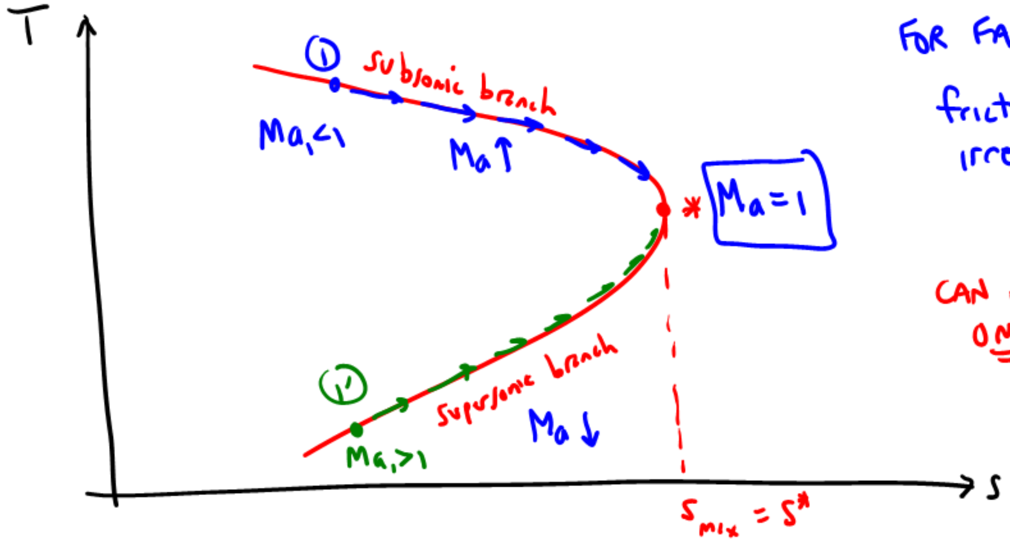


- CONS. OF MASS $\boxed{\rho_1 V_1 = \rho_2 V_2}$ (1)

- CONS. OF ENERGY

$$q = \cancel{\frac{\dot{Q}}{\dot{m}}}^{\,0} = c_p\left(T_{0_2} - T_{0_1}\right) \Rightarrow \boxed{T_{0_1} = T_{0_2}} \quad (2)$$

• COMBINE EQs (1) & (2), ideal gas law, T-ds eq & some state eqs & generate the FANNO CURVE (FANNO LINE)



FOR FANNO FLOW,

friction is an irreversibility,

s ↑

CAN MOVE TO THE RIGHT ONLY ON OUR PLOT

☆ $\boxed{\text{Ma} \rightarrow 1 \text{ (sonic or critical conditions, *) FOR EITHER SUBSONIC OR SUPERSONIC FLOW}}$

COMMENTS:

• FANNO CURVE IS SIMILAR TO THE RAYLEIGH CURVE, BUT
  - Rayleigh curve satisfies mass & momentum
  - Fanno curve satisfies mass & energy

• ENERGY EQ. DETERMINES WHERE WE LAND ON THE RAYLEIGH CURVE
• MOMENTUM EQ. DETERMINES WHERE WE LAND ON THE FANNO CURVE

• "STRANGE ZONE" IS THE ENTIRE SUBSONIC REGION IN FANNO FLOW!
  ↳ AS FRICTION ↑ T ↓ IN THE SUBSONIC BRANCH

Linear momentum equation in x-direction:

$$\sum F_x = \sum \cancel{F_{x,\text{gravity}}} + \sum F_{x,\text{pressure}} + \sum F_{x,\text{viscous}} + \sum \cancel{F_{x,\text{other}}} = \sum_{\text{out}} \overbrace{\beta}^{\beta \approx 1} \dot{m} V - \sum_{\text{in}} \overbrace{\beta}^{\beta \approx 1} \dot{m} V$$

none in x

2

$$P_1 A - P_2 A \quad - F_{\text{friction}} \quad = (1)\rho_2 V_2 A \, V_2 \; - (1)\rho_1 V_1 A \, V_1$$

÷ A & rearrange

$$\boxed{P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2 + \frac{F_{\text{friction}}}{A}} \qquad (3)$$

- Other eqs. in our toolbox
  - T-ds eqs $\quad \boxed{s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}}$
  - Ideal gas law $\quad \boxed{\frac{P_1}{\rho_1 T_1} = \frac{P_2}{\rho_2 T_2}}$
  - State eqs e.g., $\quad \boxed{\frac{T_0}{T} = 1 + \frac{k-1}{2} Ma^2}$

Summary of Equations for Fanno Flow for an Ideal Gas

Conservation laws of mass, energy, and momentum (from above notes):

mass

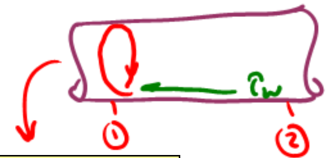
$$\rho_1 V_1 = \rho_2 V_2 \qquad \text{or} \qquad \rho V = \text{constant} \quad (1)$$

energy

$$T_{01} = T_{02} \qquad \text{or} \qquad c_p T_1 + \frac{V_1^2}{2} = c_p T_2 + \frac{V_2^2}{2} \quad (2)$$

$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2 + \frac{F_{\text{friction}}}{A} \quad (3)$$

momentum ?



How to calculate the friction force:

- Integrate shear stress along the wall, $\dfrac{F_{\text{friction}}}{A} = \dfrac{\text{perimeter}}{A} \displaystyle\int_{x_1}^{x_2} \tau_w \, dx$
- Apply the **Darcy friction factor**, $f = \dfrac{8\tau_w}{\rho V^2}$, and assume f is constant between x_1 and x_2
- Use the **Churchill equation** for f, $f = 8\left[\left(\dfrac{8}{\text{Re}}\right)^{12} + (A+B)^{-1.5}\right]^{\frac{1}{12}}$,

  where $\text{Re} = \dfrac{\rho V D_h}{\mu} = \dfrac{V D_h}{\nu}$, $A = \left\{-2.457 \cdot \ln\left[\left(\dfrac{7}{\text{Re}}\right)^{0.9} + 0.27\dfrac{\varepsilon}{D}\right]\right\}^{16}$, $B = \left(\dfrac{37530}{\text{Re}}\right)^{16}$,

  and hydraulic diameter $D_h = \dfrac{4A}{\text{perimeter}}$
- Thus, $\dfrac{F_{\text{friction}}}{A} = \dfrac{1}{2D_h} \displaystyle\int_{x_1}^{x_2} f \rho V^2 \, dx$

Plug the above equation into our momentum equation (3) and do a *lot* of algebra,

$$\frac{f}{D_h}(x_2 - x_1) = \left[-\frac{1}{k\text{Ma}^2} - \frac{k+1}{2k}\ln\left(\frac{\text{Ma}^2}{1 - \frac{k-1}{2}\text{Ma}^2}\right)\right]_{\text{Ma}_1}^{\text{Ma}_2}$$

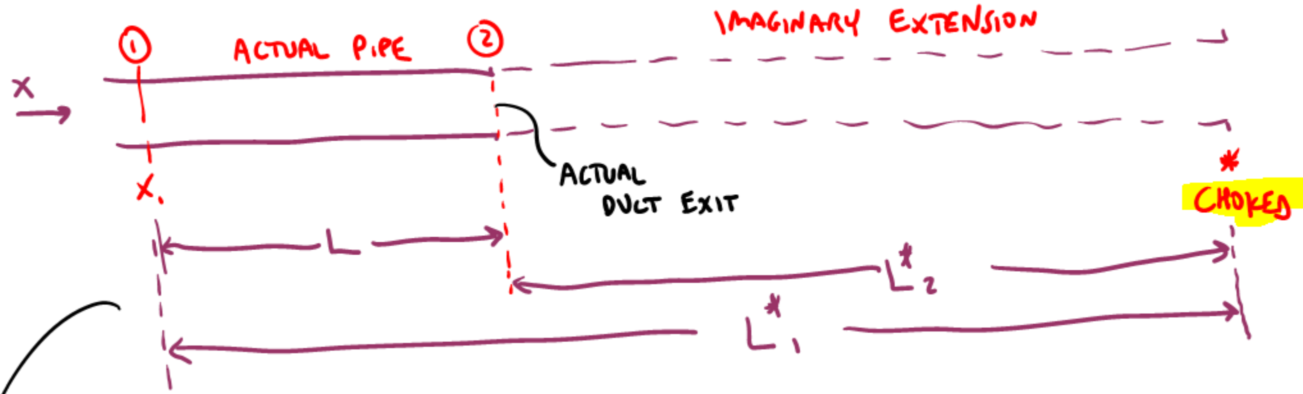
Finally, consider the **choked case**, where $\mathbf{Ma_2 = 1}$ and set $\boldsymbol{x_2 - x_1 = L^*_1}$ since the flow is choked at the exit. After some more algebra, the above equation becomes

L^*_1 is the critical length from x_1 to * (choked flow)

$$\frac{fL^*_1}{D_h} = \frac{1 - \text{Ma}_1^2}{k\text{Ma}_1^2} + \frac{k+1}{2k}\ln\left(\frac{(k+1)\text{Ma}_1^2}{2 + (k-1)\text{Ma}_1^2}\right) \quad (4)$$

Application of Equation (4) to Fanno Flow Problems

FOR KNOWN CONDITIONS AT ① (V_1, P_1, T_1, Ma_1, etc.)



DEFINE:
- L = actual duct length from ① to ②
- L^*_1 = imaginary duct length from ① to ⊛
- L^*_2 = imaginary duct length from ② to ⊛

$$L + L^*_2 = L^*_1 \longrightarrow L = L^*_1 - L^*_2$$

MULTIPLY BY $\dfrac{f}{D_h}$

$$\boxed{\frac{fL}{D_h} = \frac{fL^*_1}{D_h} - \frac{fL^*_2}{D_h}} \quad \oplus$$

- Rewrite Eq(4) @ any x & any Ma

Final "workhorse" Fanno flow equations (for *any* Mach number):

$$\boxed{\frac{fL}{D_h} = \frac{fL^*_1}{D_h} - \frac{fL^*_2}{D_h}} \quad \text{where} \quad \boxed{\frac{fL^*}{D_h} = \frac{1-\mathrm{Ma}^2}{k\mathrm{Ma}^2} + \frac{k+1}{2k}\ln\left(\frac{(k+1)\mathrm{Ma}^2}{2+(k-1)\mathrm{Ma}^2}\right)}$$

Step-by-Step Procedure to Solve Fanno Flow Problems

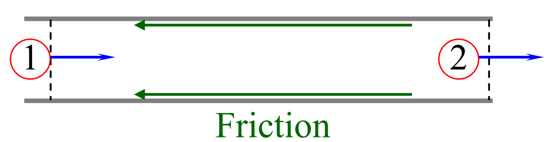
1. For known conditions at 1, duct roughness ε, hydraulic diameter D_h, and pipe length L, calculate friction factor f from the Churchill equation
2. Calculate $\dfrac{fL^*_1}{D_h}$ from workhorse equation $\boxed{\dfrac{fL^*_1}{D_h} = \dfrac{1-\mathrm{Ma}_1^{\,2}}{k\mathrm{Ma}_1^{\,2}} + \dfrac{k+1}{2k}\ln\left(\dfrac{(k+1)\mathrm{Ma}_1^{\,2}}{2+(k-1)\mathrm{Ma}_1^{\,2}}\right)}$
3. Calculate $\dfrac{fL^*_2}{D_h}$ from workhorse equation $\boxed{\dfrac{fL}{D_h} = \dfrac{fL^*_1}{D_h} - \dfrac{fL^*_2}{D_h}}$ — Solve
4. Calculate Ma_2 from workhorse equation $\boxed{\dfrac{fL^*_2}{D_h} = \dfrac{1-\mathrm{Ma}_2^{\,2}}{k\mathrm{Ma}_2^{\,2}} + \dfrac{k+1}{2k}\ln\left(\dfrac{(k+1)\mathrm{Ma}_2^{\,2}}{2+(k-1)\mathrm{Ma}_2^{\,2}}\right)}$
   IMPLICITLY ★
5. Calculate T_2 from clever use of ratios, knowing that T_0 is constant, and applying our state equation for $\dfrac{T_0}{T}$, $\boxed{T_2 = \dfrac{T_2}{T_0}\dfrac{T_0}{T_1}T_1 = \left(1+\dfrac{k-1}{2}\mathrm{Ma}_2^{\,2}\right)^{-1}\left(1+\dfrac{k-1}{2}\mathrm{Ma}_1^{\,2}\right)T_1}$
6. Knowing T_2 and Ma_2, calculate any other desired properties at location 2

Example: Fanno flow

Given:

- Air enters a 5.00-cm diameter ($D = D_h$), 27.0-m long (L) tube at 450 K (T_1), 220 kPa (P_1), and 85.0 m/s (V_1).

  The average roughness height of the inside wall of the pipe is 0.08 mm. $= \varepsilon$



- The pipe is well-insulated (→ abiabatic), but we need to be concerned about friction in the pipe since it is so long. → FANNO FLOW

To do: Estimate the temperature, pressure, velocity (speed) V_2, and Mach number at location 2.

Solution:

Assumptions and Approximations (consistent with our simplified Fanno flow analysis):

1. The air is an ideal gas, and the properties do not change with temperature or pressure.
2. The flow is steady and one-D.
3. The flow is adiabatic but friction along the tube walls is *not* negligible.
4. The Darcy friction factor f is approximated as constant based on conditions at the inlet, and the Churchill equation is used to calculate f.

Summary of inlet conditions

Known values: $V_1 = 85.0$ m/s, $T_1 = 450$ K, $P_1 = 220$ kPa,
$D_h = D = 0.0500$ m, $\varepsilon = 8.00\times10^{-5}$ m, and $L = 27.0$ m

Calculated: $\mathrm{Ma}_1 = 0.200$, $\mu_1 = 2.499\times10^{-5}$ kg/(m s) from Sutherland equation,
$\rho_1 = 1.7035$ kg/m^3 from ideal gas law

Step 1: Calculate Re and f from Churchill equation → **Re = 289,660** and **f = 0.02296**

Step 2: Calculate $\frac{fL^*_1}{D_h}$ from $\frac{fL^*_1}{D_h} = \frac{1-\text{Ma}_1^2}{k\text{Ma}_1^2} + \frac{k+1}{2k}\ln\left(\frac{(k+1)\text{Ma}_1^2}{2+(k-1)\text{Ma}_1^2}\right)$

0.200

$$\rightarrow \frac{fL^*_1}{D_h} = 14.550$$

Step 3: Calculate $\frac{fL^*_2}{D_h}$ from $\frac{fL}{D_h} = \frac{fL^*_1}{D_h} - \frac{fL^*_2}{D_h}$ → $\frac{fL^*_2}{D_h} = \frac{fL^*_1}{D_h} - \frac{fL}{D_h}$

$$\frac{fL^*_2}{D_h} = \frac{fL^*_1}{D_h} - \frac{fL}{D_h} = 14.550 - \frac{(0.02296)(27.0\text{ m})}{0.0500\text{ m}} \Rightarrow \frac{fL^*_2}{D_h} = 2.1507$$

Step 4: Calculate Ma_2 from $\frac{fL^*_2}{D_h} = \frac{1-\text{Ma}_2^2}{k\text{Ma}_2^2} + \frac{k+1}{2k}\ln\left(\frac{(k+1)\text{Ma}_2^2}{2+(k-1)\text{Ma}_2^2}\right)$

SOLVE IMPLICITLY FOR Ma_2 (I used False Position Method)

$$\rightarrow \text{Ma}_2 = 0.40902$$

Step 5: Calculate T_2 from $T_2 = \left(1+\frac{k-1}{2}\text{Ma}_2^2\right)^{-1}\left(1+\frac{k-1}{2}\text{Ma}_1^2\right)T_1$

450 K

$$T_2 = 438.91\text{ K}$$

Step 6: Calculate other properties at location 2

e.g., $c_2 = \sqrt{kRT_2} = \sqrt{(1.40)\left(287.0\ \frac{\text{m}^2}{\text{s}^2\text{K}}\right)(438.91\text{ K})} = 419.94\ \frac{\text{m}}{\text{s}}$

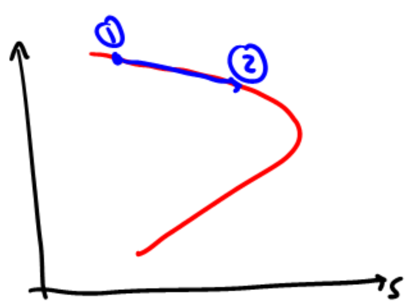
$$V_2 = c_2\text{Ma}_2 = \left(419.94\ \frac{\text{m}}{\text{s}}\right)(0.40902) = 171.76\text{ m/s}$$

$$\left.\rho_2 = \frac{\rho_1 V_1}{V_2} \ ; \ P_2 = \rho_2 R T_2 \right\} \quad P_2 = 106.19\text{ kPa}$$

FINAL ANSWERS: (3 digits)

$$\boxed{\begin{aligned} T_2 &= 439.\text{ K} \\ P_2 &= 106.\text{ kPa} \\ V_2 &= 172.\text{ m/s} \\ \text{Ma}_2 &= 0.409 \end{aligned}}$$

T ↓
P ↓
V ↑
Ma ↑



(since we are on subsonic branch)

Additional verification of our assumptions

Let's see if Darcy friction factor remains nearly constant by looking at the Moody chart:

Moody Chart: [Figure from Cengel and Cimbala, *Fluid Mechanics: Fundamentals and Applications*, Ed. 4.]

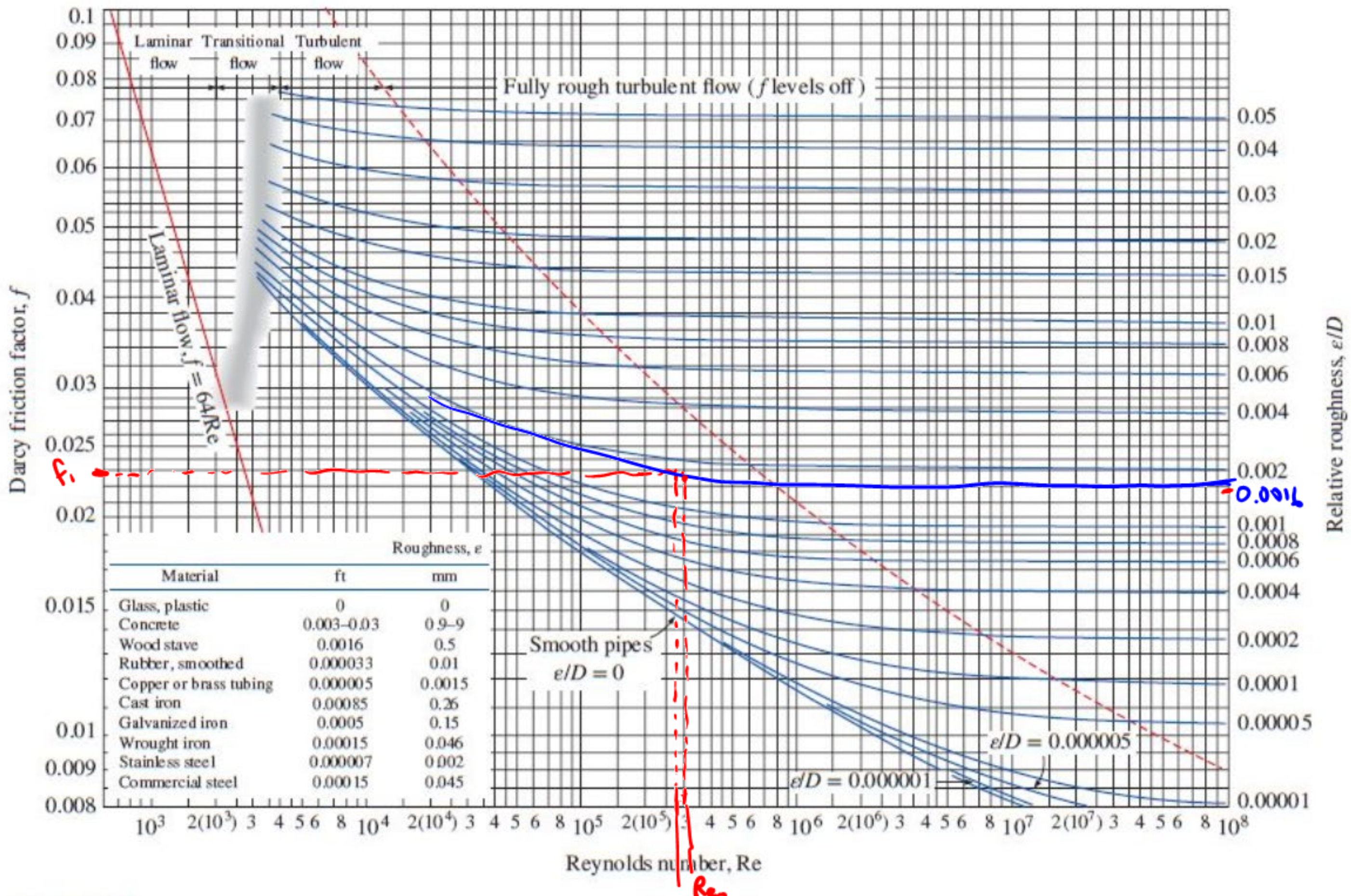


FIGURE A–12

The Moody chart for the friction factor for fully developed flow in circular pipes for use in the head loss relation $h_L = f\frac{L}{D}\frac{V^2}{2g}$. Friction factors in the turbulent flow are evaluated from the Colebrook equation $\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$.

@ $Re_1 = \underline{289{,}660}$; $\frac{\varepsilon}{D_h}$ 0.0016 $\Rightarrow f_1 = 0.02296$

@ $Re_2 = \underline{300{,}706}$ 〃 $\Rightarrow f_2 = 0.002293$

$\Big\}$ $\underline{f \approx \text{const}}$

VERY GOOD APPROXIMATION !

One zero should not be here