Today, we will:

- Discuss the motion and deformation of fluid particles
- Discuss linear strain, shear strain, and the strain rate tensor
- Begin to discuss the Reynolds Transport Theorem (RTT)

C. Other Kinematic Descriptions (Section 4-4)

1. Motion and deformation of fluid particles

Vorticity and Rotationality (Section 4-5)

The **vorticity vector** is defined as the **curl of the velocity vector**,

Greek letter zeta

$$\vec{\zeta} = \vec{\nabla} \times \vec{V}$$

It turns out that **vorticity is equal to twice the angular velocity of a fluid particle**,

$$\vec{\zeta} = 2\vec{\omega}$$

Thus, ***vorticity is a measure of rotation of a fluid particle***.

if $\vec{\zeta} = 0$, the flow is irrotational

if $\vec{\zeta} \neq 0$, the flow is rotational

Vorticity vector in Cartesian coordinates:

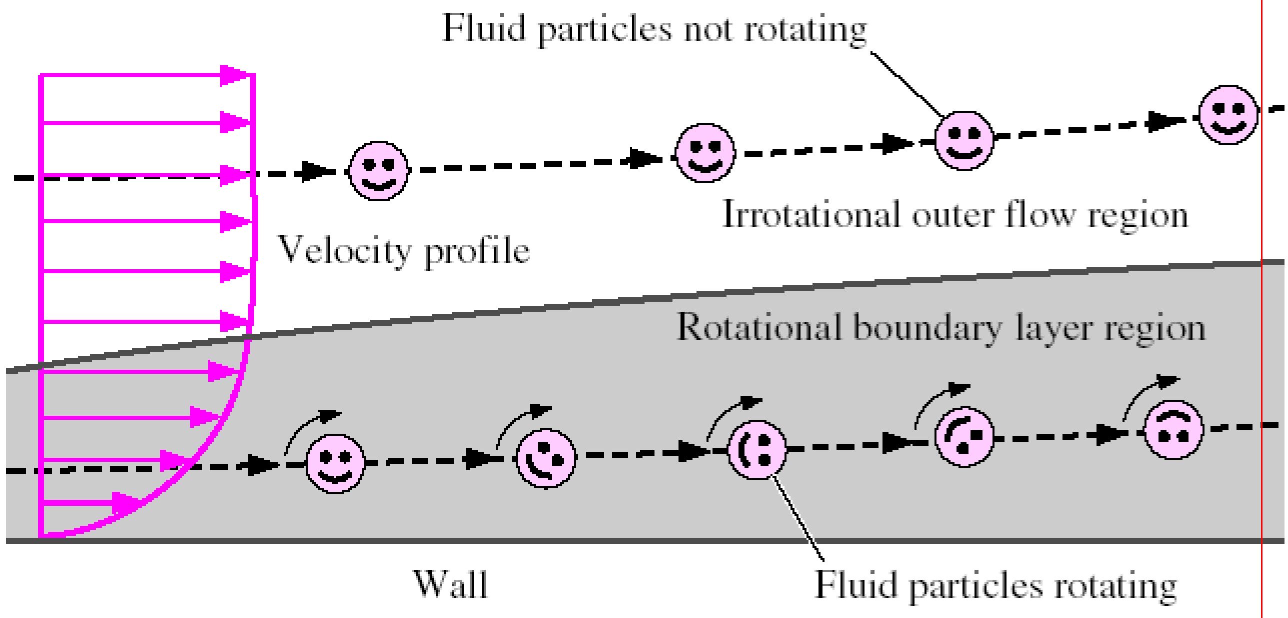
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} \qquad \textbf{(4–30)}$$

Vorticity vector in cylindrical coordinates:

$$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z \qquad \textbf{(4–32)}$$

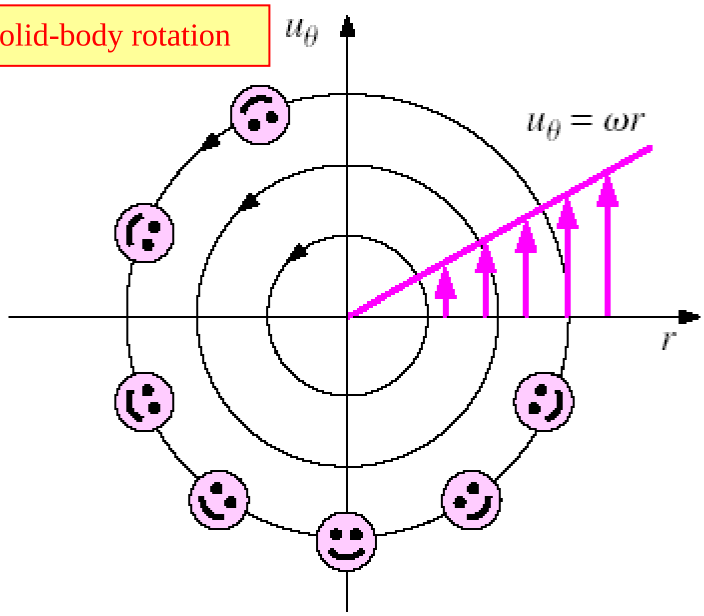
Examples:

1. Inside a **boundary layer**, where viscous forces are important, the flow in this region is *rotational* ($\vec{\zeta} \neq 0$). However, outside the boundary layer, where viscous forces are not important, the flow in this region is *irrotational* ($\vec{\zeta} = 0$).



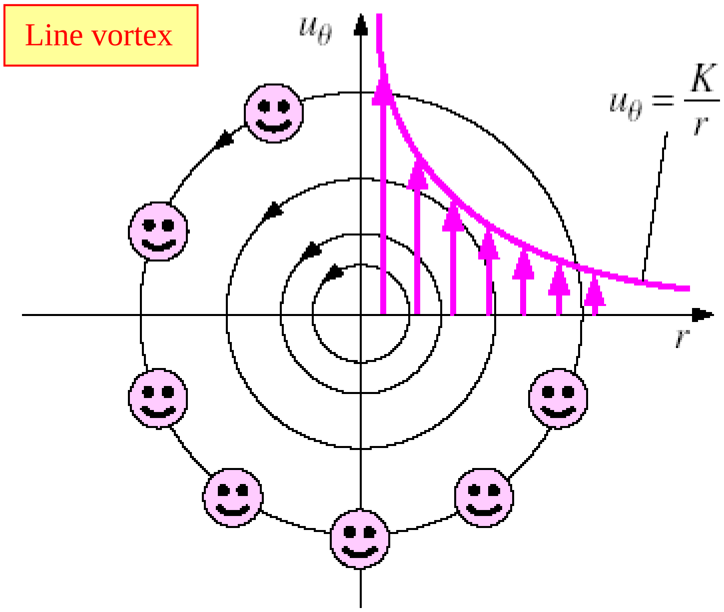
2. A **solid-body rotation** (rigid-body rotation) flow is *rotational* ($\vec{\zeta} \neq 0$). In fact, since vorticity is equal to twice the angular velocity, $\vec{\zeta} = 2\vec{\omega}$ *everywhere* in the flow field. Fluid particles rotate as they revolve around the center of the flow. This is analogous to a merry-go-round or a roundabout.

Solid-body rotation



3. A **line vortex** flow, however, is *irrotational* ($\vec{\zeta} = 0$), and fluid particles do not rotate, even though they revolve around the center of the flow. This is analogous to a Ferris wheel.

Line vortex



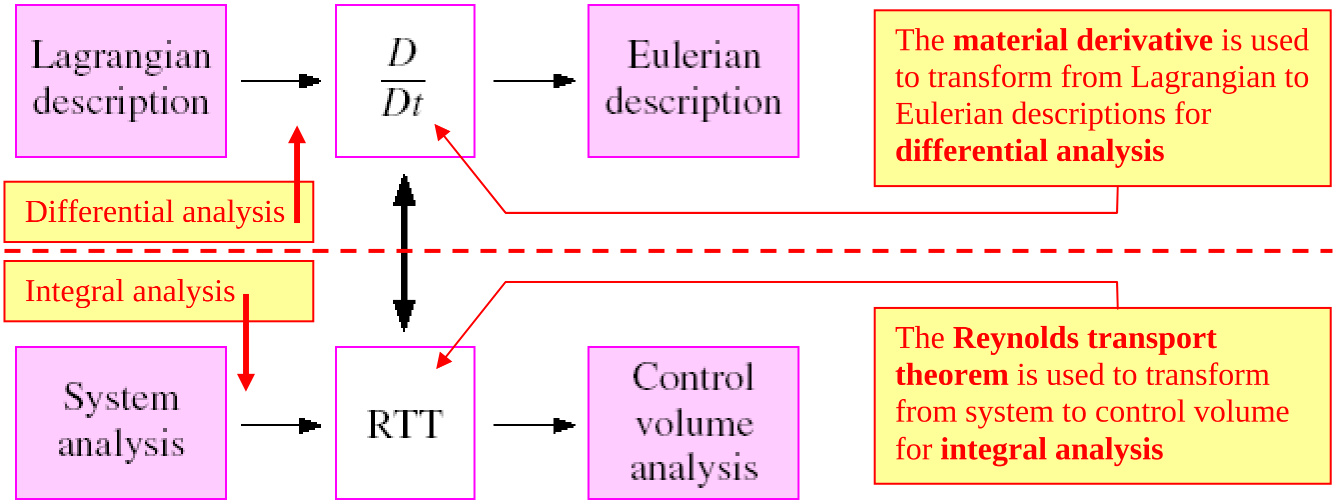
See text for details and calculations.

The Reynolds Transport Theorem (RTT) (Section 4-6)

Recall from Thermodynamics:

- A ***system*** [also called a ***closed system***] is a quantity of matter of fixed identity. *No mass can cross a system boundary*.
- A ***control volume*** [also called an ***open system***] is a region in space chosen for study. *Mass **can** cross a control surface* (the surface of the control volume).
- The fundamental conservation laws (conservation of mass, energy, and momentum) *apply directly to systems*.
- However, in most fluid mechanics problems, ***control volume analysis is preferred over system analysis*** (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).
- Therefore, we need to *transform the conservation laws from a system to a control volume*. This is accomplished with the ***Reynolds transport theorem (RTT)***.

There is a direct **analogy** between the transformation from Lagrangian to Eulerian descriptions (for differential analysis using infinitesimally small fluid elements) and the transformation from systems to control volumes (for integral analysis using large, finite flow fields):



The **material derivative** is used to transform from Lagrangian to Eulerian descriptions for **differential analysis**

The **Reynolds transport theorem** is used to transform from system to control volume for **integral analysis**

FIGURE 4-64

In both cases, the fundamental laws of physics (conservation laws) are known in the analysis on the left (Lagrangian or system), and must be transformed so as to be useful in the analysis on the right (Eulerian or control volume).

Another way to think about the RTT is that it is a *link* between the system approach and the control volume approach:

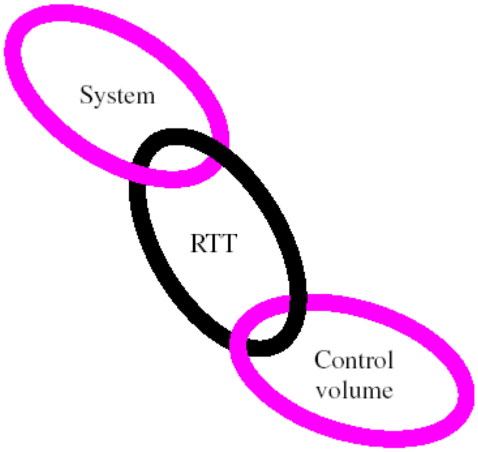


FIGURE 4-54

See text for detailed derivation of the RTT. Here are some highlights:

- Let B represent any extensive property (like mass, energy, or momentum).
- Let b be the corresponding intensive property, i.e., $b = B/m$ (property B per unit mass).
- Our goal is to find a relationship between B_{sys} or b_{sys} (property of the system, for which we know the conservation laws) and B_{CV} or b_{CV} (property of the control volume, which we prefer to use in our analysis).
- The results are shown below in various forms:

For **fixed** (non-moving and non-deforming) control volumes,

RTT, fixed CV:

$$\frac{dB_{\text{sys}}}{dt} = \boxed{\frac{d}{dt}\int_{\text{CV}} \rho b \, dV} + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n} \, dA \qquad \text{(4–41)}$$

Alternate RTT, fixed CV:

$$\frac{dB_{\text{sys}}}{dt} = \boxed{\int_{\text{CV}} \frac{\partial}{\partial t}(\rho b) \, dV} + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n} \, dA \qquad \text{(4–42)}$$

Since the control volume is *fixed*, the order of integration or differentiation does not matter, i.e. $\frac{d}{dt}\int_{\text{CV}} \ldots$ is the same as $\int_{\text{CV}} \frac{\partial}{\partial t} \ldots$. Thus, the two circled quantities above are *equivalent* for a fixed control volume.

For **nonfixed** (moving and/or deforming) control volumes,

RTT, nonfixed CV:

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b \, dV + \int_{\text{CS}} \rho b \boxed{\vec{V}_r} \, \vec{n} \, dA \qquad \textbf{(4–44)}$$

Note that we replace $\vec{V}$ by $\vec{V}_r$ in this version of the RTT for a moving and/or deforming control volume.

where **Error! Objects cannot be created from editing field codes.** is the **relative velocity**, i.e., the velocity of the fluid *relative to the control surface* (which may be moving or deforming),

Relative velocity:

$$\vec{V}_r = \vec{V} - \vec{V}_{\text{CS}} \qquad \textbf{(4–43)}$$

We can also switch the order of the time derivative and the integral in the first term on the right, but only if we use the *absolute* (rather than the relative) velocity in the second term on the right, i.e.,

Alternate RTT, nonfixed CV:

$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n} \, dA \qquad \textbf{(4–45)}$$

Comparing Eqs. 4-45 and 4-42, we see that they are identical. Thus, the most general form of the RTT that *applies to both fixed and non-fixed control volumes* is

General RTT, nonfixed CV:

$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n} dA \qquad \textbf{(4–53)}$$

Even though this equation is most general, it is often easier *in practice* to use Eq. 4-44 for moving and/or deforming (non-fixed) control volumes because the algebra is easier.

Simplifications:

- For **steady** flow, the volume integral drops out. In terms of relative velocity,

RTT, steady flow:

$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CS}} \rho b \vec{V}_r \cdot \vec{n} \, dA$$

- For control volumes where there are **well-defined inlets and outlets**, the control surface integral can be simplified, avoiding cumbersome integrations,

Approximate RTT for well-defined inlets and outlets:

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b \, dV + \sum_{\text{out}} \underbrace{\rho_{\text{avg}} b_{\text{avg}} V_{r,\,\text{avg}} A}_{\text{for each outlet}} - \sum_{\text{in}} \underbrace{\rho_{\text{avg}} b_{\text{avg}} V_{r,\text{avg}} A}_{\text{for each inlet}} \qquad \textbf{(4–48)}$$

Note that the above equation is *approximate*, and may not always be accurate.