Today, we will:

- Derive and discuss the linear momentum equation for a control volume (Chapter 6)
- Discuss the momentum flux correction factor, β
- Discuss all the various forces acting on a control volume
- Do some example problems – Linear momentum equation for a control volume

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E. The Linear Momentum Equation for a Control Volume (Chapter 6)

1. Equations and Definitions

Recall from the RTT at the end of Chapter 4,

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \int_{CS} \rho \vec{V} \rho \left(\vec{V} \cdot \vec{n}\right) dA$$

Total force (***vector***) acting on the control volume

Rate of change of linear momentum inside the control volume

Net rate of linear momentum flow out of the control volume

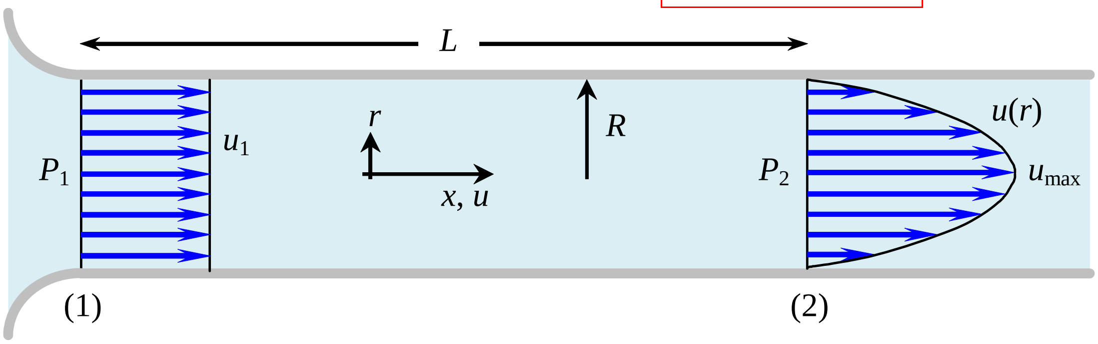
Use *relative* velocity $\vec{V}_r$ here if we have a moving or deforming control volume

4. Examples

Example: Friction force in a pipe

Given: Consider steady, laminar, incompressible, axisymmetric flow of a liquid in a pipe as sketched. At the inlet (1) there is a nice bell mouth, and the velocity is nearly uniform (except for a very thin boundary layer, not shown).

- At (1), $u = u_1$ = constant, $v = 0$, and $w = 0$. P_1 is measured.
- At (2), the flow is fully developed and parabolic: $u_2 = u_{max}\left(1-\frac{r^2}{R^2}\right)$. P_2 is measured.



To do: Calculate the total friction force acting on the fluid by the pipe wall from 1 to 2.

Solution:

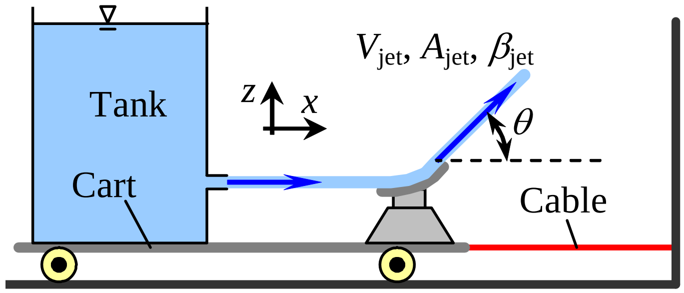
• First step:

- Now use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

Example: Tension in a cable

Given: A cart with frictionless wheels and a large tank shoots water at a deflector plate, turning it by angle θ as sketched. The cart tries to move to the left, but a cable prevents it from doing so. At the exit of the deflector, the water jet area A_{jet}, its average velocity V_{jet}, and its momentum flux correction factor β_{jet} are known.



To do: Calculate the tension T in the cable.

Solution:

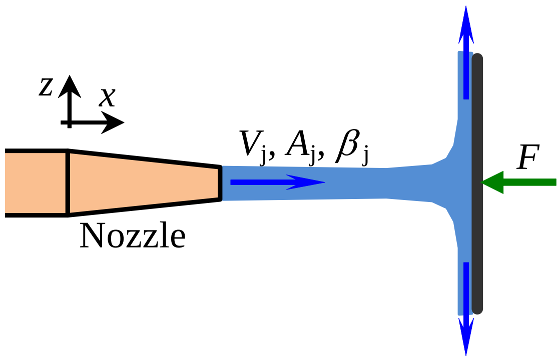
- First step:
- Second step: Use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

Example: Force imparted by a water jet hitting a stationary plate

Given: A horizontal water jet of area A_j, average velocity V_j, and momentum flux correction factor β_j impinges normal to a stationary vertical flat plate.

To do: Calculate the horizontal force F required to keep the plate from moving.



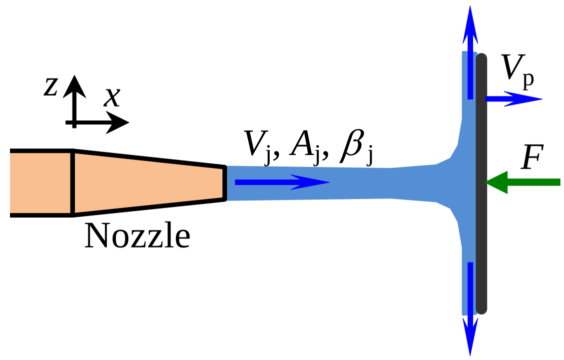
Solution:

- First step:
- Second step: Use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

Example: Force imparted by a water jet hitting a *moving* plate

Given: A horizontal water jet of area A_j, average velocity V_j, and momentum flux correction factor β_j impinges normal to a *moving* vertical flat plate. The plate moves to the right at speed V_p.



To do: Calculate the horizontal force F required to keep the plate moving at constant speed V_p.

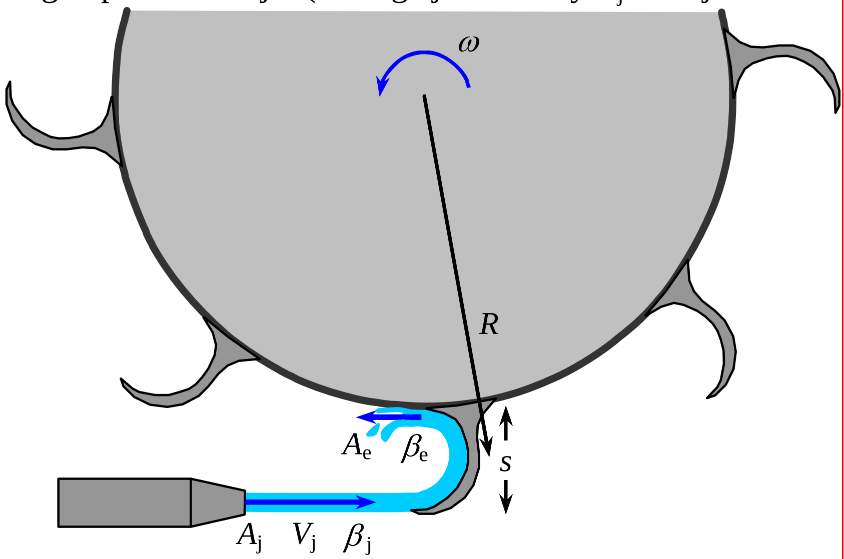
Solution:

- First step:
- Second step: Use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

Example: Force on a bucket of a Pelton-type (impulse) hydroturbine

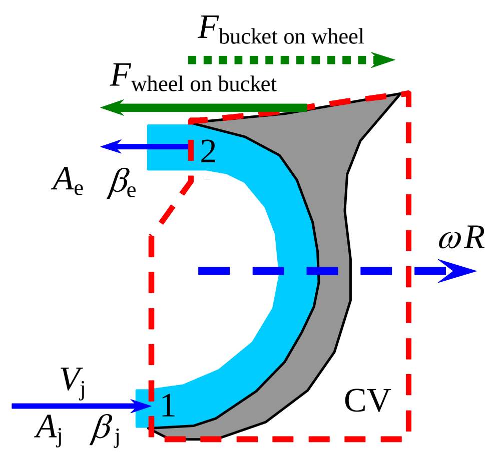
Given: An impulse turbine is driven by a high-speed water jet (average jet velocity V_j over jet area A_j, with momentum flux correction factor β_j) that impinges on turning buckets attached to a turbine wheel as shown. The turbine wheel rotates at angular velocity ω, and is horizontal; therefore, gravity effects are not important in this problem. (The view in the sketch is from the top.) The turning buckets turn the water approximately 180 degrees, and the water exits the bucket over exit cross-sectional area A_e with exit momentum flux correction factor β_e. For simplicity, we approximate that the bucket dimension s is much smaller than turbine wheel radius R ($s << R$).



(**a**) **To do**: Calculate the force of the bucket on the turbine wheel, $F_{\text{bucket on wheel}}$, at the instant in time when the bucket is in the position shown.

(**b**) **To do**: Calculate the power delivered to the turbine wheel.

Solution: We choose a control volume surrounding the bucket, cutting through the water jet at the inlet to the bucket, and cutting through the water exiting the bucket. Note that this is a *moving control volume*, moving to the right at speed ωR. We also cut through the welded joint between the bucket and the turbine wheel, where the force $F_{\text{bucket on wheel}}$ is to be calculated. Because of Newton's third law, the force acting *on the control volume* at this location is equal in magnitude, but opposite in direction, and we call it $F_{\text{bucket on wheel}}$.



Since the pressure through an incompressible jet exposed to atmospheric air is equal to P_{atm}, the pressure at the inlet (1) is equal to P_{atm}, and the pressure at the exit (2) is also equal to P_{atm}.

Solution to be completed in class.

- Use the *x*-component of the steady linear momentum equation for a moving CV,

$$\sum \vec{F}_x = \sum \vec{F}_{x,\text{ gravity}} + \sum \vec{F}_{x,\text{ pressure}} + \sum \vec{F}_{x,\text{ viscous}} + \sum \vec{F}_{x,\text{ other}} = \sum_{\text{out}} \beta \dot{m} u_r - \sum_{\text{in}} \beta \dot{m} u_r$$