**Today, we will**:

- Do another example – the stream function.
- Discuss the differential equation for momentum in fluid flow: **The Navier-Stokes eq**.
- Do some example problems – Navier-Stokes equation

**Derivation of the Navier-Stokes Equation**
**(Section 9-5, Çengel and Cimbala)**

We begin with the general differential equation for conservation of linear momentum, i.e., ***Cauchy's equation***, which is valid for any kind of fluid,

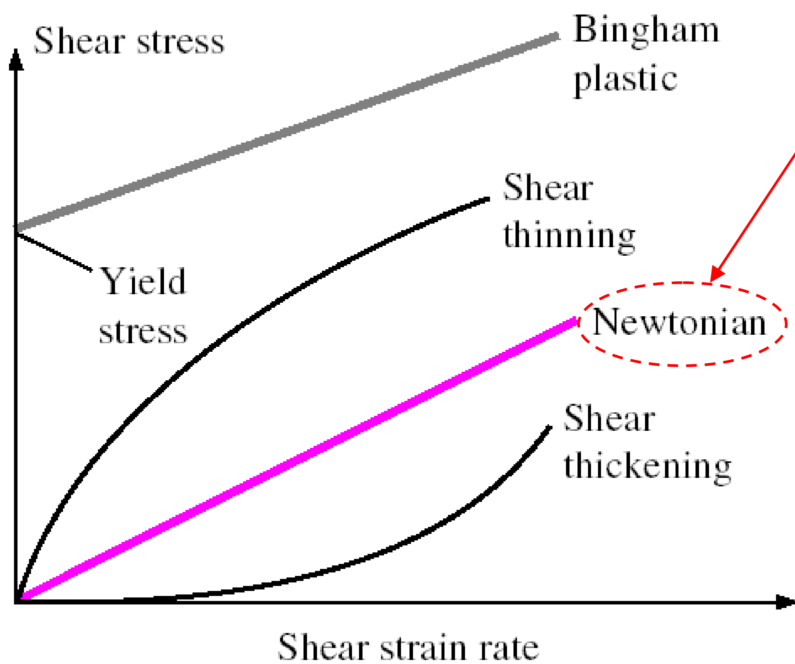
Cauchy's equation:

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij} \qquad (9\text{–}50)$$

Stress tensor ($\sigma_{ij}$)

The problem is that the stress tensor $\sigma_{ij}$ needs to be written in terms of the primary unknowns in the problem in order for Cauchy's equation to be useful to us. The equations that relate $\sigma_{ij}$ to other variables in the problem – velocity, pressure, and fluid properties – are called ***constitutive equations***. There are different constitutive equations for different kinds of fluids.

Types of fluids:




For ***Newtonian fluids**, the shear stress is linearly proportional to the shear strain rate.*

Examples of Newtonian fluids: water, air, oil, gasoline, most other common fluids.

**FIGURE 9–37**
Rheological behavior of fluids—shear stress as a function of shear strain rate.

Some examples of non-Newtonian fluids:

- Paint (***shear thinning*** or ***pseudo-plastic***)
- Toothpaste (***Bingham plastic***)
- Quicksand (***shear thickening*** or ***dilatant***).

We consider only Newtonian fluids in this course.

For ***Newtonian fluids*** (see text for derivation), it turns out that

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu \frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\mu \frac{\partial w}{\partial z} \end{pmatrix} \quad \textbf{(9–57)}$$

We have achieved our goal of writing $\sigma_{ij}$ in terms of pressure $P$, velocity components $u$, $v$ , and $w$, and fluid viscosity $\mu$.

Now we plug this expression for the stress tensor $\sigma_{ij}$ into Cauchy's equation. The result is the famous ***Navier-Stokes equation***, shown here for incompressible flow.

*Incompressible Navier–Stokes equation*:

Navier-Stokes equation:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{V} \quad \textbf{(9–60)}$$

To solve fluid flow problems, we need both the continuity equation and the Navier-Stokes equation. Since it is a vector equation, the Navier-Stokes equation is usually split into three components in order to solve fluid flow problems. In Cartesian coordinates,

*Incompressible continuity equation*:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \textbf{(9–61a)}$$

*x-component of the incompressible Navier–Stokes equation*:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \textbf{(9–61b)}$$

*y-component of the incompressible Navier–Stokes equation*:

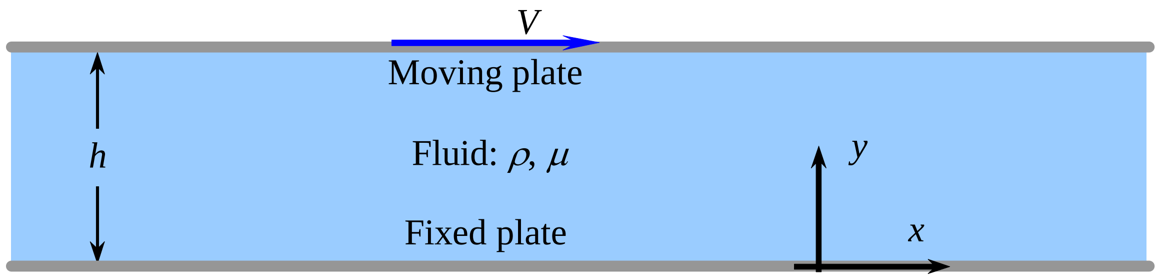
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad \textbf{(9–61c)}$$

*z-component of the incompressible Navier–Stokes equation*:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad \textbf{(9–61d)}$$

# Example Problem – Exact Solution for Couette Flow

**Given**: Steady, incompressible, laminar flow in the *x*-*y* plane between two infinite parallel plates.




Assumptions and approximations:

1. The flow is steady [$\partial/\partial t$ of anything = 0].
2. The flow is two-dimensional in the *x*-*y* plane [$\partial/\partial z$ of anything = 0, $w = 0$].
3. Gravity effects are negligible or ignored.
4. The flow is fully developed [$\partial/\partial x$ of any velocity = 0 – velocity does not change with *x*].
5. Pressure is constant everywhere.

**To do**: Calculate the velocity field.

**Solution**: [to be done in class]