

**Today, we will:**

- Begin discussion about Chapter 10 – Approximate solutions of the N-S equation
- Show how to nondimensionalize the equations of motion
- Discuss creeping flow (flow at very low Reynolds number)

**VIII. APPROXIMATE SOLUTIONS OF THE NAVIER-STOKES EQUATION****A. Introduction**

We have three ways to solve the differential equations of fluid flow:

1. Analytically (Chapter 9) [solve exactly, but only for very simple problems]
2. Numerically (Chapter 15) [use CFD on a computer to solve for thousands of cells]
3. Approximately (Chapter 10) [ignore some terms in the N-S equation, then solve]

**B. Nondimensionalization of the Equations of Motion**

# Nondimensionalization of the Navier-Stokes Equation

(Section 10-2, Çengel and Cimbala)

## Nondimensionalization:

We begin with the differential equation for conservation of linear momentum for a Newtonian fluid, i.e., the *Navier-Stokes equation*. For incompressible flow,

$$\rho \frac{D\vec{V}}{Dt} = \rho \left[ \frac{\partial \vec{V}}{\partial t} + \left( \vec{V} \cdot \vec{\nabla} \right) \vec{V} \right] = -\vec{\nabla}P + \rho \vec{g} + \mu \nabla^2 \vec{V} \quad (10-2)$$

Equation 10-2 is *dimensional*, and each variable or property ( $\rho$ ,  $\vec{V}$ ,  $t$ ,  $\mu$ , etc.) is also *dimensional*. What are the primary dimensions (in terms of {m}, {L}, {t}, {T}, etc) of each term in this equation?

Answer: {      }

To nondimensionalize Eq. 10-2, we choose *scaling parameters* as follows:

**TABLE 10-1**

Scaling parameters used to nondimensionalize the continuity and momentum equations, along with their primary dimensions

Scaling Parameter	Description	Primary Dimensions
$L$	Characteristic length	{L}
$V$	Characteristic speed	{L t <sup>-1</sup> }
$f$	Characteristic frequency	{t <sup>-1</sup> }
$P_0 - P_\infty$	Reference pressure difference	{m L <sup>-1</sup> t <sup>-2</sup> }
$g$	Gravitational acceleration	{L t <sup>-2</sup> }

We define *nondimensional variables*, using the scaling parameters in Table 10-1:

$$\begin{aligned}
 t^* &= ft & \vec{x}^* &= \frac{\vec{x}}{L} & \vec{V}^* &= \frac{\vec{V}}{V} \\
 P^* &= \frac{P - P_\infty}{P_0 - P_\infty} & \vec{g}^* &= \frac{\vec{g}}{g} & \vec{\nabla}^* &= L \vec{\nabla}
 \end{aligned} \quad (10-3)$$

To plug Eqs. 10-3 into Eq. 10-2, we need to first rearrange the equations in terms of the dimensional variables, i.e.,

$$\begin{aligned}
 t &= \frac{1}{f} t^* & \vec{x} &= L \vec{x}^* & \vec{V} &= V \vec{V}^* \\
 P &= P_\infty + (P_0 - P_\infty) P^* & \vec{g} &= g \vec{g}^* & \vec{\nabla} &= \frac{1}{L} \vec{\nabla}^*
 \end{aligned}$$

Now we substitute all of the above into Eq. 10-2 to obtain

$$\rho V f \frac{\partial \vec{V}^*}{\partial t^*} + \frac{\rho V^2}{L} (\vec{V}^* \cdot \vec{\nabla}^*) \vec{V}^* = -\frac{P_0 - P_\infty}{L} \vec{\nabla}^* P^* + \rho g \vec{g}^* + \frac{\mu V}{L^2} \nabla^{*2} \vec{V}^*$$

Every additive term in the above equation has primary dimensions  $\{m^1 L^{-2} t^{-2}\}$ . To nondimensionalize the equation, we multiply every term by constant  $L/(\rho V^2)$ , which has primary dimensions  $\{m^{-1} L^2 t^2\}$ , so that the dimensions cancel. After some rearrangement,

$$\left[\frac{fL}{V}\right] \frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^* \cdot \vec{\nabla}^*) \vec{V}^* = -\left[\frac{P_0 - P_\infty}{\rho V^2}\right] \vec{\nabla}^* P^* + \left[\frac{gL}{V^2}\right] \vec{g}^* + \left[\frac{\mu}{\rho VL}\right] \nabla^{*2} \vec{V}^* \quad (10-5)$$

Strouhal number, where

$$St = \frac{fL}{V}$$

Euler number, where

$$Eu = \frac{P_0 - P_\infty}{\rho V^2}$$

Inverse of Froude number squared, where

$$Fr = \frac{V}{\sqrt{gL}}$$

Inverse of Reynolds number, where

$$Re = \frac{\rho VL}{\mu}$$

Thus, Eq. 10-5 can therefore be written as

Navier-Stokes equation in nondimensional form:

$$[St] \frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^* \cdot \vec{\nabla}^*) \vec{V}^* = -[Eu] \vec{\nabla}^* P^* + \left[\frac{1}{Fr^2}\right] \vec{g}^* + \left[\frac{1}{Re}\right] \nabla^{*2} \vec{V}^* \quad (10-6)$$

### Nondimensionalization vs. Normalization:

Equation 10-6 above is *nondimensional*, but not necessarily *normalized*. What is the difference?

- **Nondimensionalization** concerns only the *dimensions* of the equation – we can use *any* value of scaling parameters  $L$ ,  $V$ , etc., and we always end up with Eq. 10-6.
- **Normalization** is more restrictive than nondimensionalization. To *normalize* the equation, we must choose scaling parameters  $L$ ,  $V$ , etc. that are appropriate for the flow being analyzed, such that ***all nondimensional variables*** ( $t^*$ ,  $\vec{V}^*$ ,  $P^*$ , etc.) ***in Eq. 10-6 are of order of magnitude unity***. In other words, their minimum and maximum values are reasonably close to 1.0 (e.g.,  $-6 < P^* < 3$ , or  $0 < P^* < 11$ , but *not*  $0 < P^* < 0.001$ , or  $-200 < P^* < 500$ ). We express the normalization as follows:

$$t^* \sim 1, \quad \vec{x}^* \sim 1, \quad \vec{V}^* \sim 1, \quad P^* \sim 1, \quad \vec{g}^* \sim 1, \quad \vec{\nabla}^* \sim 1$$

If we have properly normalized the Navier-Stokes equation, we can compare the relative importance of various terms in the equation by comparing the *relative magnitudes* of the nondimensional parameters  $St$ ,  $Eu$ ,  $Fr$ , and  $Re$ .