Today, we will:

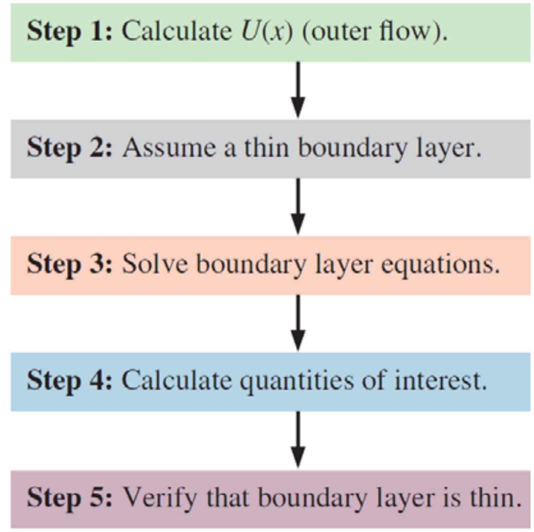
- Do a BL example, boundary layer on a flat plate aligned with the flow

Review: The Boundary Layer Equations

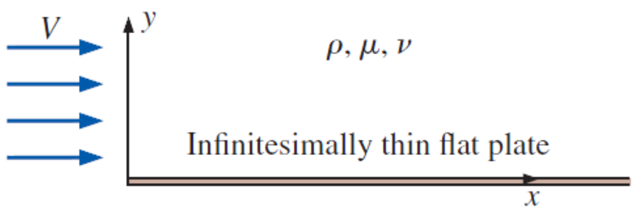
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Review: The Boundary Layer Procedure

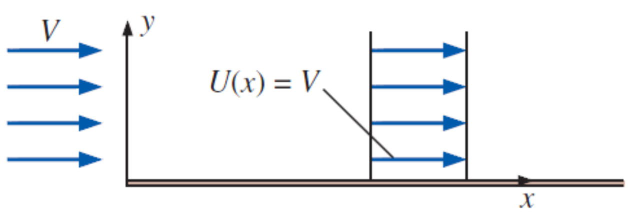


Example: The Laminar Flat Plate Boundary Layer



We go through the steps of the boundary layer procedure:

- **Step 1**: The outer flow is ***U*(*x*) = *U* = *V* = constant**. In other words, the outer flow is simply a uniform stream of constant velocity.
- **Step 2**: A very thin boundary layer is assumed (so thin that it does not affect the outer flow). In other words, the outer flow does not even know that the boundary layer is there.



- **Step 3**: The boundary layer equations must be solved; they reduce to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

There are four required boundary conditions,

$$u = 0 \quad \text{at } y = 0 \qquad u = U \quad \text{as } y \to \infty$$

$$v = 0 \quad \text{at } y = 0 \qquad u = U \quad \text{for all } y \text{ at } x = 0$$

This equation set was first solved by **P. R. H. Blasius** in 1908 – numerically, but *by hand*!

Blasius introduced a **similarity variable** η that combines independent variables x and y into one nondimensional independent variable,

Similarity variable

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

and he solved for a nondimensionalized form of the x-component of velocity,

$$f' = \frac{u}{U} = \text{function of } \eta$$

The similarity solution is f' as a function of η.

The key here is that ***one single similarity velocity profile holds for any x-location along the flat plate***. In other words, the velocity profile shape is the same ("similar") at any location, but it is merely *stretched vertically* as the boundary layer grows down the plate. This is illustrated in Fig. 10-98 in the text.

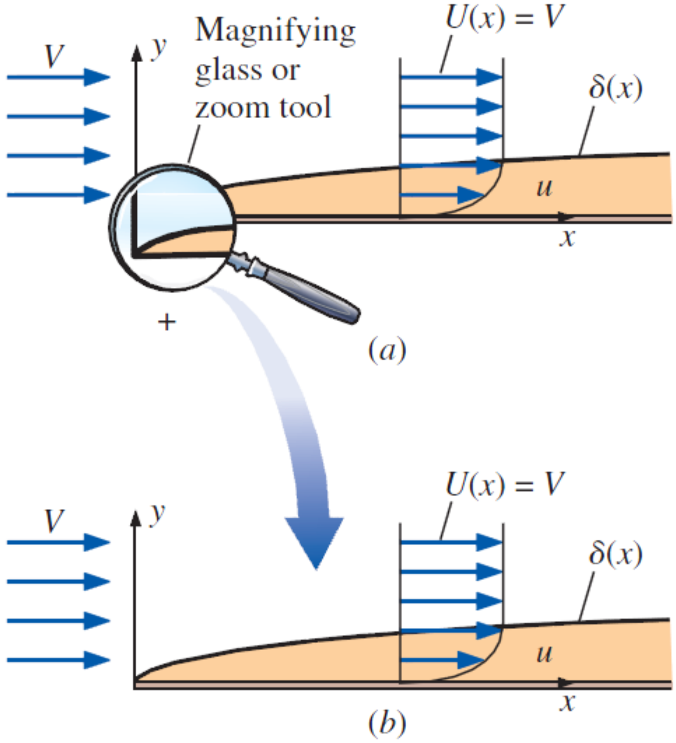
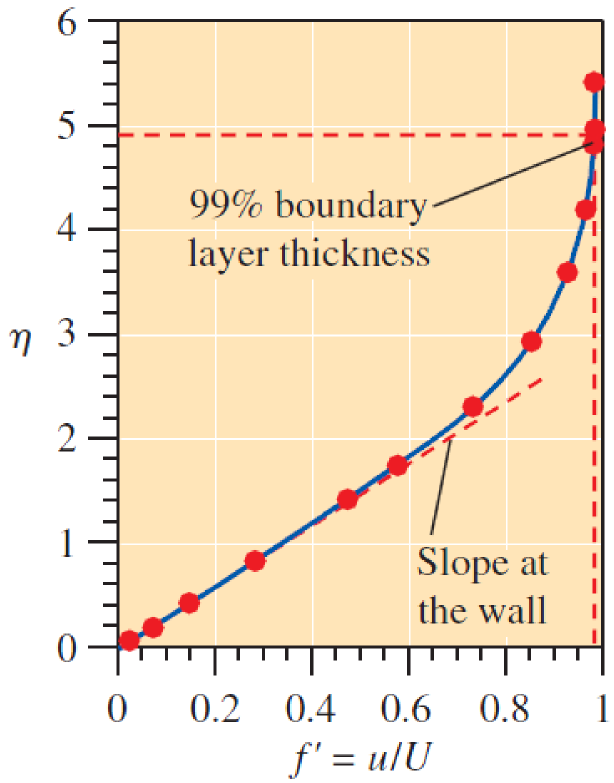


FIGURE 10–98
A useful result of the similarity assumption is that the flow looks the same (is *similar*) regardless of how far we zoom in or out; (*a*) view from a distance, as a person might see, (*b*) close-up view, as an ant might see.

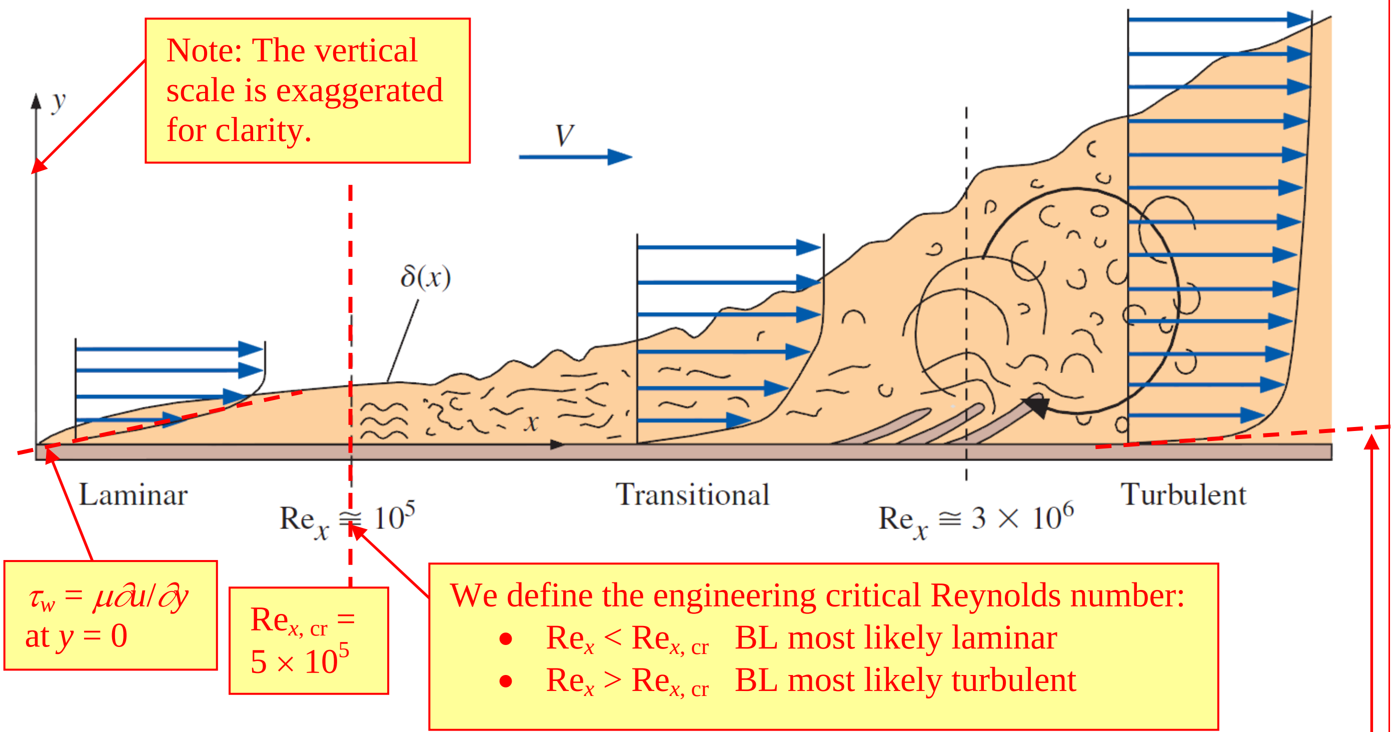
The similarity solution itself is tabulated in Table 10-3, and is plotted in Fig. 10-99.

FIGURE 10–99
The Blasius profile in similarity variables for the boundary layer growing on a semi-infinite flat plate. Experimental data (circles) are at $Re_x = 3.64 \times 10^5$.
From Panton (1996).

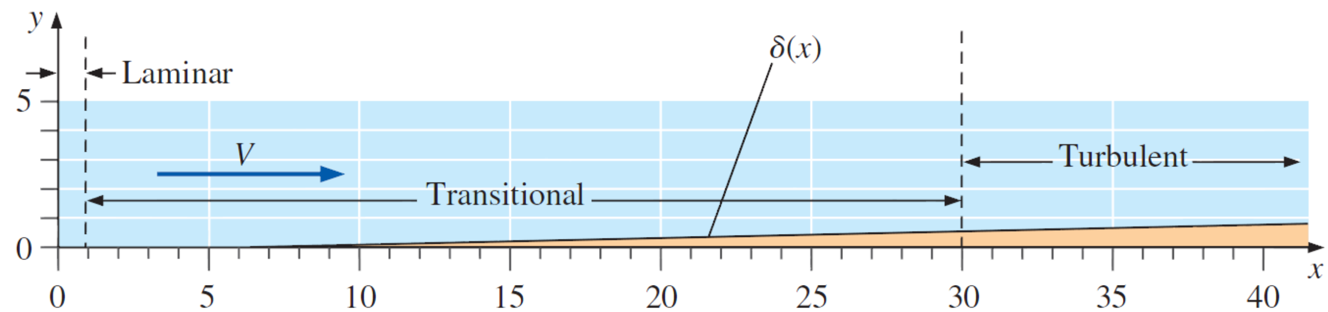


This one velocity profile, plotted in nondimensional form as above, applies at *any* *x*- location in the boundary layer.

The Turbulent Flat Plate Boundary Layer (Section 10-6, Çengel and Cimbala)



Here is what the actual BL looks like to scale:



The turbulent flat plate boundary layer velocity profile:

The time-averaged turbulent flat plate (zero pressure gradient) boundary layer velocity profile is much *fuller* than the laminar flat plate boundary layer profile, and therefore has a larger slope $\partial u/\partial y$ at the wall, leading to greater skin friction drag along the wall.

There are three common empirical relationships for the turbulent flat plate boundary layer velocity profile:

- **The log law**:

The log law:

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{y u_*}{\nu} + B \qquad \textbf{(10–83)}$$

where

Friction velocity:

$$u_* = \sqrt{\frac{\tau_w}{\rho}} \qquad \textbf{(10–84)}$$

- **Spalding's law of the wall**:

$$\frac{yu_*}{\nu} = \frac{u}{u_*} + e^{-\kappa B}\left[e^{\kappa(u/u_*)} - 1 - \kappa(u/u_*) - \frac{[\kappa(u/u_*)]^2}{2} - \frac{[\kappa(u/u_*)]^3}{6}\right] \quad \textbf{(10–85)}$$

- **The one-seventh-power law**:

$$\frac{u}{U} \cong \left(\frac{y}{\delta}\right)^{1/7} \quad \text{for } y \le \delta, \quad \rightarrow \quad \frac{u}{U} \cong 1 \quad \text{for } y > \delta \quad \textbf{(10–82)}$$

Quantities of interest for the turbulent flat plate boundary layer:

Just as we did for the laminar (Blasius) flat plate boundary layer, we can use these expressions for the velocity profile to estimate quantities of interest, such as the 99% boundary layer thickness δ, the displacement thickness δ^*, the local skin friction coefficient $C_{f,x}$, etc. These are summarized in Table 10-4 in the text.

Column (b) expressions are generally preferred for engineering analysis.

TABLE 10–4

Summary of expressions for laminar and turbulent boundary layers on a smooth flat plate aligned parallel to a uniform stream*

Property	Laminar	(a) Turbulent(†)	(b) Turbulent(‡)
Boundary layer thickness	$\frac{\delta}{x} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta}{x} \cong \frac{0.16}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta}{x} \cong \frac{0.38}{(\mathrm{Re}_x)^{1/5}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta^*}{x} \cong \frac{0.020}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta^*}{x} \cong \frac{0.048}{(\mathrm{Re}_x)^{1/5}}$
Momentum thickness	$\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$\frac{\theta}{x} \cong \frac{0.016}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\theta}{x} \cong \frac{0.037}{(\mathrm{Re}_x)^{1/5}}$
Local skin friction coefficient	$C_{f,x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$C_{f,x} \cong \frac{0.027}{(\mathrm{Re}_x)^{1/7}}$	$C_{f,x} \cong \frac{0.059}{(\mathrm{Re}_x)^{1/5}}$

Note that $C_{f,x}$ is the *local* skin friction coefficient, applied at only *one* value of x.

To these we add the integrated ***average skin friction coefficients*** for *one side* of a flat plate of length L, noting that C_f applies to the entire plate from $x = 0$ to $x = L$ (see Chapter 11):

Laminar: $$C_f = \frac{1.33}{\mathrm{Re}_L^{1/2}} \quad \mathrm{Re}_L \lesssim 5 \times 10^5 \quad \textbf{(11–19)}$$

Turbulent: $$C_f = \frac{0.074}{\mathrm{Re}_L^{1/5}} \quad 5 \times 10^5 \lesssim \mathrm{Re}_L \lesssim 10^7 \quad \textbf{(11–20)}$$

For cases in which the laminar portion of the plate is taken into consideration, we use:

$$C_f = \frac{0.074}{\mathrm{Re}_L^{1/5}} - \frac{1742}{\mathrm{Re}_L} \qquad 5 \times 10^5 \lesssim \mathrm{Re}_L \lesssim 10^7 \qquad \textbf{(11–22)}$$

Turbulent flat plate boundary layers with wall roughness:

Finally, all of the above are for *smooth* flat plates. However, if the plate is *rough*, the average skin friction coefficient C_f increases with roughness ε. This is similar to the situation in pipe flows, in which Darcy friction factor f increases with pipe wall roughness.

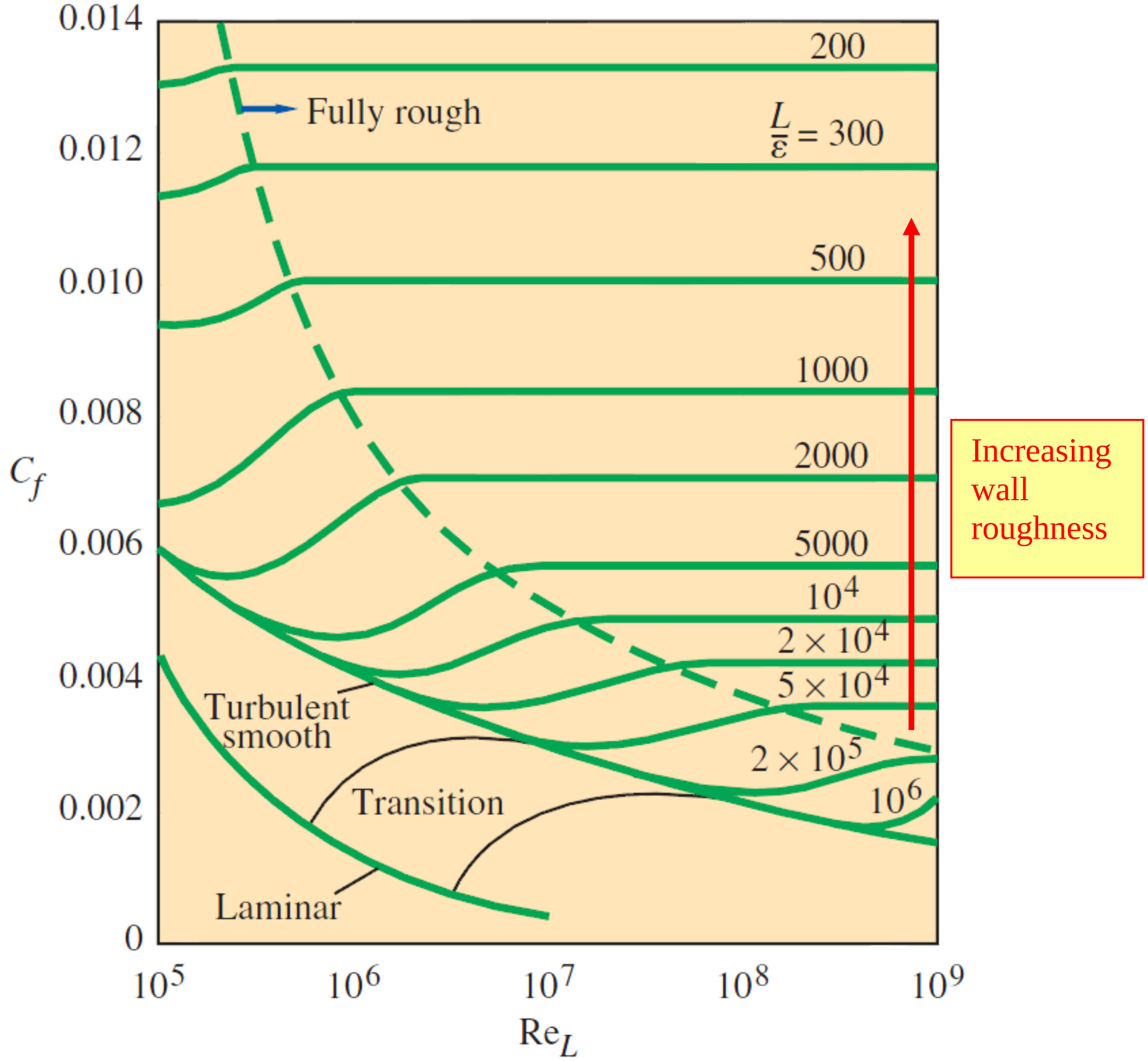


FIGURE 11–31

Friction coefficient for parallel flow over smooth and rough flat plates.

Just as with pipe flows, at high enough Reynolds numbers, the boundary layer becomes "fully rough". For a ***fully rough flat plate turbulent boundary layer*** with average wall roughness height ε,

Fully rough turbulent regime:

$$C_f = \left(1.89 - 1.62 \log \frac{\varepsilon}{L}\right)^{-2.5} \qquad \textbf{(11–23)}$$

This equation represents the flat portions of Fig. 11-31 that are labeled "Fully rough".