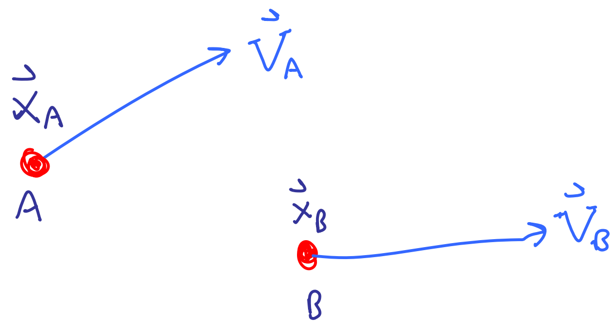
Today, we will:

- Begin Chapter 4 – FLUID KINEMATICS → describe motion w/o discussing the physics
- Discuss the material acceleration and the material derivative, and show examples
- Discuss various kinds of flow patterns and flow visualization techniques
- Begin to discuss other kinematic properties (motion and deformation of fluid particles)

III. FLUID KINEMATICS

A. Descriptions of Fluid Flow – there are two ways to describe fluid flow:

1. Lagrangian description → follow & keep track of individual fluid particles as functions of time

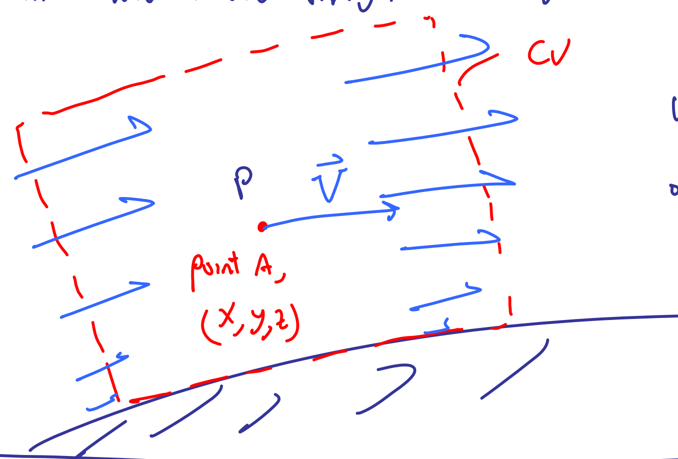


↓ define $\vec{x}_A(t)$
$\vec{x}_B(t)$
& etc.

Great for billiard balls, simple physics experiments

2. Eulerian description → preferred description

↓ We identify a region in the flow & call it a control volume & watch the fluid move through that region



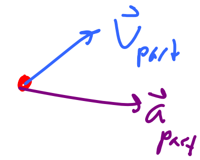
We use field variables or flow field variables

e.g. $\boxed{\vec{V} = \vec{V}(x, y, z, t), \quad P = P(x, y, z, t)}$

Problem → Physics eqs are known for particles (Lagrangian). We need to convert from Lagrangian to Eulerian.

3. Acceleration field and material derivative

Derivation of Material Acceleration (Section 4-1)



Goal → transform $\vec a_{part}$ into a field variable $\vec a(x, y, z, t)$

Acceleration of a fluid particle:

$$\boxed{\vec a_{\text{particle}} = \frac{d\vec V_{\text{particle}}}{dt}} \qquad (4\text{–}6)$$

This is a *Lagrangian* description of the acceleration of a fluid particle.

However, at any instant in time t, the velocity of the particle is the same as the local value of the velocity *field* at the location ($x_{\text{particle}}(t)$, $y_{\text{particle}}(t)$, $z_{\text{particle}}(t)$) of the particle, since the fluid particle moves with the fluid by definition. In other words, $\vec V_{\text{particle}}(t) \equiv \vec V(x_{\text{particle}}(t), y_{\text{particle}}(t), z_{\text{particle}}(t), t)$. To take the time derivative in Eq. 4–6, we must therefore use the *chain rule*, since the dependent variable ($\vec V$) is a function of *four* independent variables (x_{particle}, y_{particle}, z_{particle}, and t),

Recall the ***chain rule***: If f is a function of two variables, t and some variable s which is itself also a function of t, then we take the total derivative of f with respect to t as follows:

$f = f(s(t), t)$ →

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial s}\frac{ds}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial s}\frac{ds}{dt}$$

Now let's apply this chain rule to the time derivative of the fluid particle's velocity:

Note that from the Lagrangian description (following a fluid particle, x_{particle} is a function of time, since the particle's location changes with time. Thus, $x_{\text{particle}} = x_{\text{particle}}(t)$. Similarly, $y_{\text{particle}} = y_{\text{particle}}(t)$ and $z_{\text{particle}} = z_{\text{particle}}(t)$.

Thus, the acceleration of a fluid particle is calculated using the chain rule as follows:

(u, v, w) = velocity comp.

$$\vec a_{\text{particle}} = \frac{d\vec V_{\text{particle}}}{dt} = \frac{d\vec V}{dt} = \frac{d\vec V(x_{\text{particle}}, y_{\text{particle}}, z_{\text{particle}}, t)}{dt}$$

$$= \frac{\partial \vec V}{\partial t}\frac{dt}{dt} + \frac{\partial \vec V}{\partial x_{\text{particle}}}\frac{dx_{\text{particle}}}{dt} + \frac{\partial \vec V}{\partial y_{\text{particle}}}\frac{dy_{\text{particle}}}{dt} + \frac{\partial \vec V}{\partial z_{\text{particle}}}\frac{dz_{\text{particle}}}{dt} \qquad (4\text{–}7)$$

$dt/dt = 1$ | $dx_{\text{particle}}/dt = u$ | $dy_{\text{particle}}/dt = v$ | $dz_{\text{particle}}/dt = w$

⇓

Or, finally,

$$\boxed{\vec a_{\text{particle}}(x, y, z, t) = \frac{d\vec V}{dt} = \frac{\partial \vec V}{\partial t} + u\frac{\partial \vec V}{\partial x} + v\frac{\partial \vec V}{\partial y} + w\frac{\partial \vec V}{\partial z}} \qquad (4\text{–}8)$$

Argue that at the instant in time under consideration,

$$\boxed{\underbrace{\vec{a}(x,y,z,t)}_{\text{accel. field at this location \& time}} = \underbrace{\vec{a}_{particle}(x,y,z,t)}_{\text{accel. of a fluid particle at this same location \& time}}}$$

Eulerian — Lagrangian

∴ The accel. field in the Eulerian description is

$$\boxed{\vec{a}(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{(\vec{V}\cdot\vec{\nabla})\vec{V}}}$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

we use special notation in fluids for this:

$$\frac{d\vec{V}}{dt} = \frac{D\vec{V}}{Dt}$$

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}} \equiv \text{Material Derivative}$$

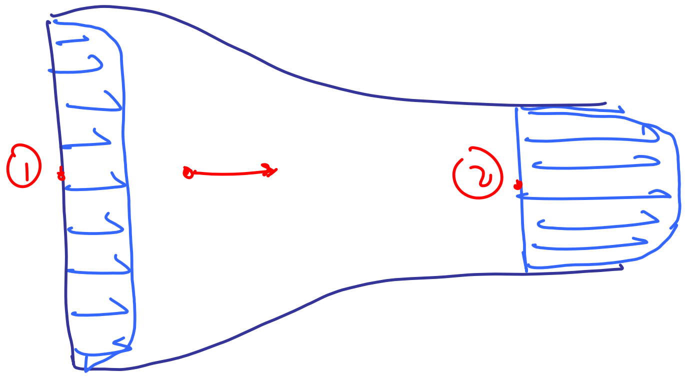
(following a fluid particle)

local part (unsteady part)

advective or convective part (due to movement of the particle to another location in the flow)

Even in a steady flow, $\frac{D}{Dt}$ can be non-zero.

Physical Example: Steady, converging duct flow



$V_2 > V_1$

$u_2 > u_1$

$\therefore \vec{a} \neq 0$

a particle moving from ① to ② accelerates

Mathematical example:

Given: Steady velocity field $\vec{V}(x,y) = 3x\,\vec{i} - 3y\,\vec{j}$

To do: Calculate steady acceleration field $\vec{a}(x,y)$

Soln: Tempting to say $\vec{a} = \frac{d\vec{V}}{dt} = 0$ since $\vec{V} \neq$ fnc. of time

✗ WRONG!

Proper soln: $\vec{a} = \frac{D\vec{V}}{Dt} = \cancel{\frac{\partial \vec{V}}{\partial t}} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$

$0 \qquad (3x)(3\vec{i}) + (-3y)(-3\vec{j}) + 0$

$$\boxed{\vec{a} = 9x\,\vec{i} + 9y\,\vec{j}}$$

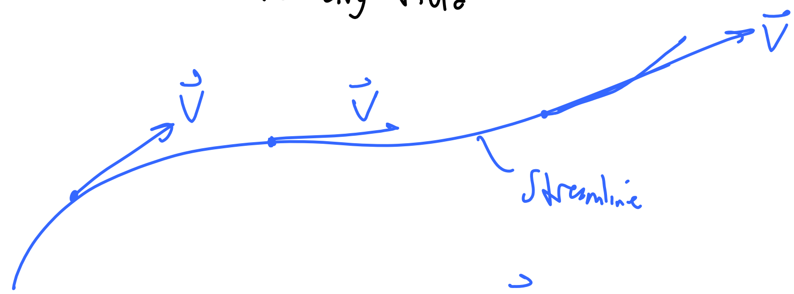
Answer $\vec{a} \neq 0$!

★ The acceleration field is non-zero even though this is a steady flow !

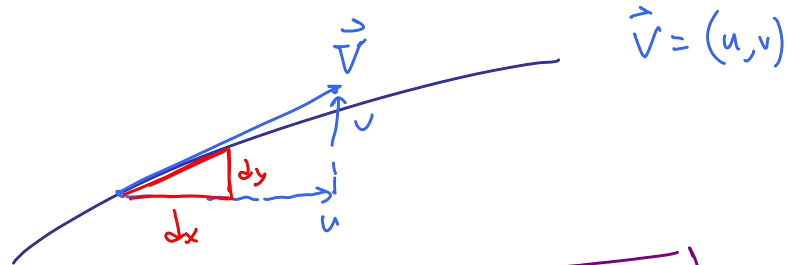
B. Flow Patterns & Flow Visualization

1. Streamlines, Pathlines, Streaklines, & Timelines

a. Streamline $\equiv$ a curve everywhere tangent to the local velocity field



In 2D,



2 similar triangles. $\therefore$ $$\boxed{\left.\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u}} \quad \star$$

Example:

Given: $\vec{V} = \underbrace{3x}_{u}\,\vec{i} \; \underbrace{- 3y}_{v}\,\vec{j}$

To do: Calc. eq. for the streamlines

Soln:
$$\frac{dy}{dx} = \frac{v}{u} \quad \text{along a streamline}$$

$$\frac{dy}{dx} = \frac{-3y}{3x} = \frac{-y}{x} \qquad\qquad \frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \quad \longrightarrow \quad \ln y = -\ln x + C$$

call $C = -\ln C$

$$\ln y = -[\ln x + \ln C]$$

$$\ln y = -\ln(cx) = \ln(cx)^{-1}$$

recall, $\ln a + \ln b = \ln(ab)$

recall $b \ln a = \ln a^b$

$e^{()}$ both sides $\rightarrow$ $y = \frac{1}{cx}$ or $\boxed{y = \frac{C_2}{x}}$ eq of streamline

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b. Pathline $\equiv$ the path traveled by a marked fluid particle over some time period.

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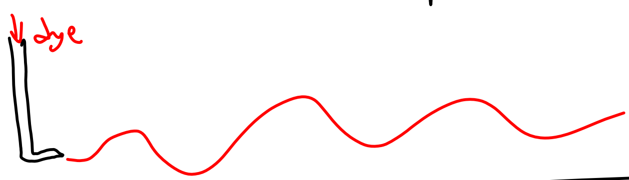
c. Streakline $\equiv$ locus of fluid particles introduced at a point



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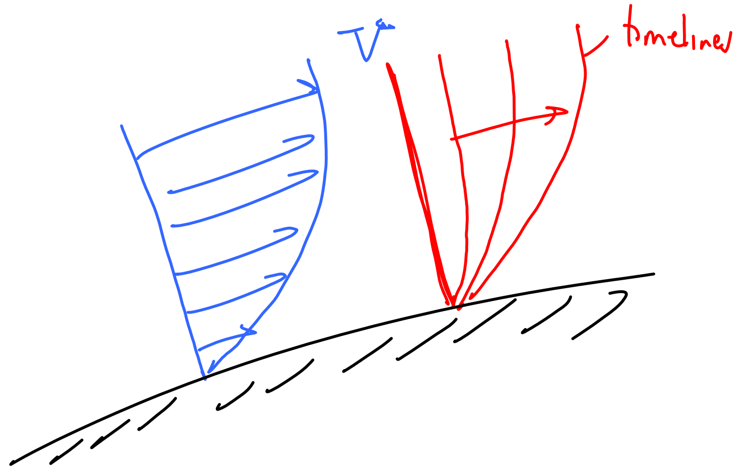
d. timeline $\equiv$ a set of adjacent fluid particles that were marked at the same time

e.g. in a boundary layer



For a steady flow, streamlines, streaklines, & pathlines are all coincident (i.e. follow the same curve)

For an unsteady flow, all of them are different.

SEE ANIMATED IMAGES ON THE WEBSITE, WEEK 3 FOR A SIMPLE CASE