Today, we will:

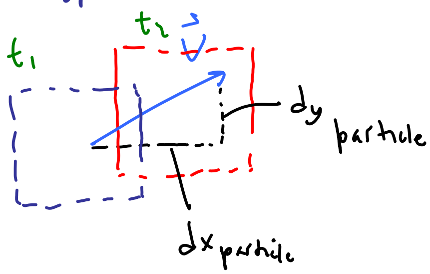
- Discuss the motion and deformation of fluid particles
- Discuss linear strain, shear strain, and the strain rate tensor
- Begin to discuss the Reynolds Transport Theorem (RTT)

C. Other Kinematic Descriptions (Section 4-4)

1. Motion and deformation of fluid particles

There are 4 fundamental types of fluid motion & deformation

a. Translation



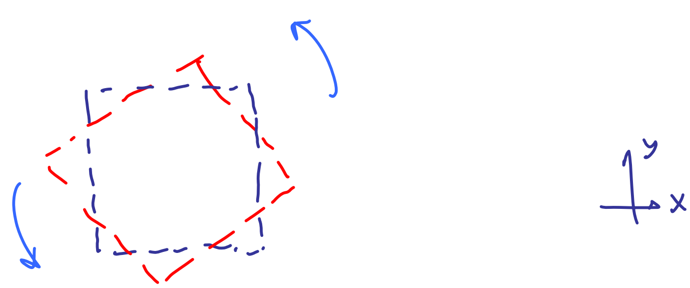
- In fluid mechanics, we prefer to work with rates of motion & deformation
- The rate of translation $= \vec{V} = \frac{dx_{particle}}{dt}\vec{i} + \frac{dy_{particle}}{dt}\vec{j} + \frac{dz_{particle}}{dt}\vec{k}$

$u \qquad v \qquad w$

$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$

= the rate of translation

b. rotation & vorticity



See text for derivation (lots of trig)

rate of rotation = the angular velocity vector

$$\boxed{\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}}$$

Define $\vec{\zeta}$ = VORTICITY $\equiv 2\vec{\omega}$, shows the rotation rate in a fluid flow

Vorticity and Rotationality (Section 4-5)

The **vorticity vector** is defined as the **curl of the velocity vector**,

Greek letter zeta

$$\boxed{\vec{\zeta} = \vec{\nabla} \times \vec{V}}$$

It turns out that **vorticity is equal to twice the angular velocity of a fluid particle**,

$$\boxed{\vec{\zeta} = 2\vec{\omega}}$$

Thus, ***vorticity is a measure of rotation of a fluid particle***.

if $\vec{\zeta} = 0$, the flow is irrotational

if $\vec{\zeta} \neq 0$, the flow is rotational

Vorticity vector in Cartesian coordinates:

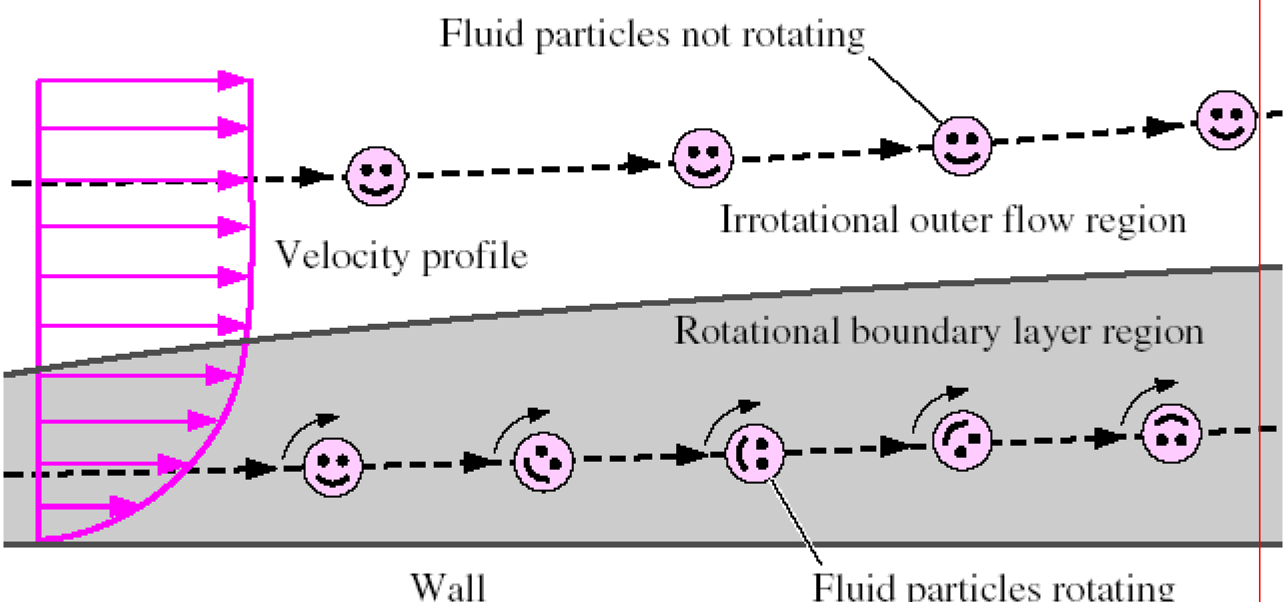
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} \tag{4–30}$$

Vorticity vector in cylindrical coordinates:

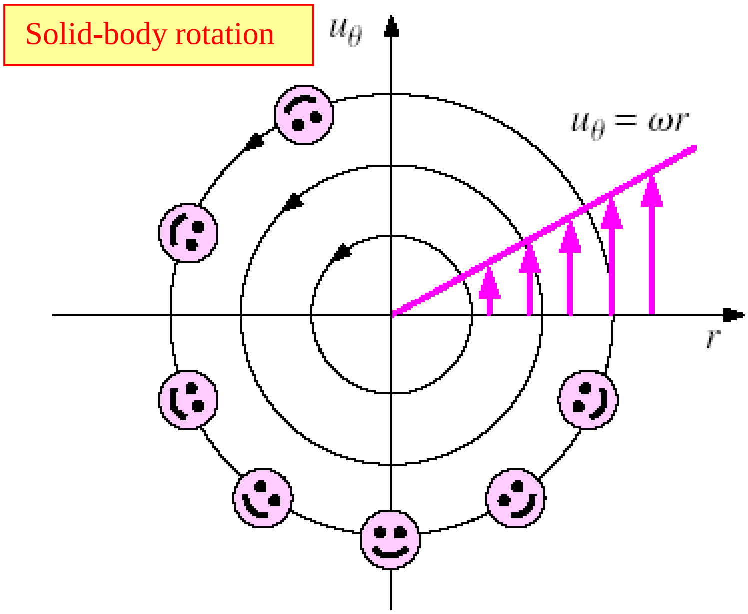
$$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z \tag{4–32}$$

Examples:

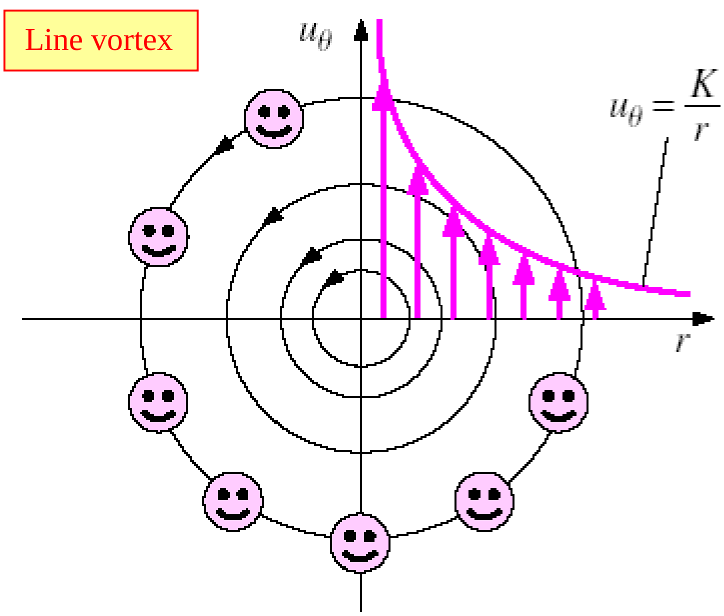
1. Inside a **boundary layer**, where viscous forces are important, the flow in this region is *rotational* ($\vec{\zeta} \neq 0$). However, outside the boundary layer, where viscous forces are not important, the flow in this region is *irrotational* ($\vec{\zeta} = 0$).



2. A **solid-body rotation** (rigid-body rotation) flow is *rotational* ($\vec{\zeta} \neq 0$). In fact, since vorticity is equal to twice the angular velocity, $\vec{\zeta} = 2\vec{\omega}$ *everywhere* in the flow field. Fluid particles rotate as they revolve around the center of the flow. This is analogous to a merry-go-round or a roundabout.



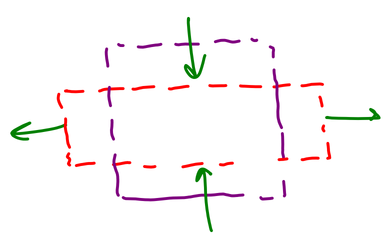
3. A **line vortex** flow, however, is *irrotational* ($\vec{\zeta} = 0$), and fluid particles do not rotate, even though they revolve around the center of the flow. This is analogous to a Ferris wheel.



See text for details and calculations.

c. Linear strain (stretching or compressing)

due to normal stresses on the particle



See text for derivation

Define strain = increase in length per unit length

We are interested in the rate of linear strain = linear strain rate

We get Linear strain rate $\boxed{\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}}$

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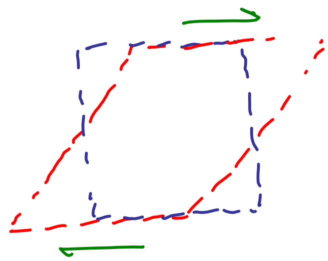
1. Shear strain



Due to shear stresses on the fluid particle

See text for derivation

rate of shear strain = shear strain rate

$$\boxed{\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)}$$

2. Strain Rate Tensor → Combine linear strain rate & shear strain rate into one 9-component 3x3 matrix (a tensor)

$$\boxed{\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}}$$

EXAMPLE: Given: $\vec{V} = (u,v) = 3x\,\vec{i} - 3y\,\vec{j}$ $\qquad W=0$ (it is 2-D)

To do: calculate rates of translation & deformation

a) rate of translation = velocity vector $\Rightarrow$ $\boxed{\begin{aligned} u &= 3x \\ v &= -3y \end{aligned}}$

b) rate of rotation = angular velocity vector

$$\vec{\omega} = (\cancel{\ \ })\,\vec{i} + (\cancel{\ \ })\,\vec{j} + (\cancel{\ \ })\,\vec{k}$$

0 (2-D) $\qquad$ 0 (2-D) $\qquad$ 0 $\quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ $\quad$ 0 $\quad$ 0

$$\boxed{\vec{\omega} = 0, \quad \vec{\zeta} = 2\vec{\omega} = 0}$$

This flow is irrotational

$u = 3x, \quad v = -3y$

c) linear strain rate

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 3$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -3$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

Notice: $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 3 - 3 + 0 = 0$

When $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0$, the flow is incompressible

Volumetric strain rate $\equiv \frac{1}{\forall}\frac{D\forall}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

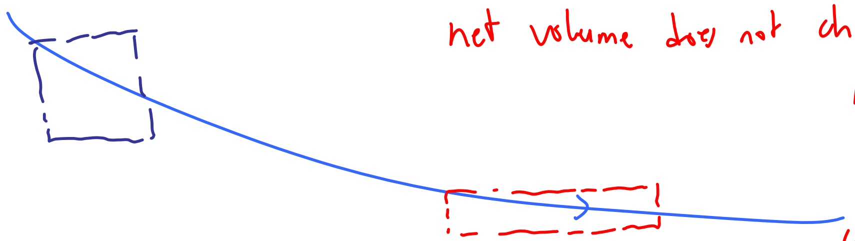
= rate of change of volume per unit volume following a fluid particle

if $\frac{1}{\forall}\frac{D\forall}{Dt} = 0$, the flow is incompressible

d) shear strain rate $\qquad u = 3x \quad v = -3y$

$$\varepsilon_{xy} = \frac{1}{2}\left(\underbrace{\cancel{\frac{\partial u}{\partial y}}}_{0} + \underbrace{\cancel{\frac{\partial v}{\partial x}}}_{0}\right) = 0 \qquad \underbrace{\varepsilon_{zx} = 0}_{(2\text{-D})}, \quad \underbrace{\varepsilon_{yz} = 0}_{(2\text{-D})}$$

$\therefore$ There is no shear strain in this flow



net volume does not change & it has no shear

Strain rate tensor

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \cancel{\varepsilon_{xz}} \\ \varepsilon_{yx} & \varepsilon_{yy} & \cancel{\varepsilon_{yz}} \\ \cancel{\varepsilon_{zx}} & \cancel{\varepsilon_{zy}} & \cancel{\varepsilon_{zz}} \end{pmatrix} \quad (2\text{-D})$$

$$\boxed{\varepsilon_{ij} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

The x & y axes here are principal axes (all off-diagonal terms are zero)

CONCLUSION – THIS FLOW IS INCOMPRESSIBLE & IRROTATIONAL.

☆ – FLUID PARTICLES STRETCH IN x, SHRINK in y, but do not distort angularly (no shear strain)