Today, we will:

- Discuss the Reynolds Transport Theorem (RTT)
- Show how the RTT applies to the conservation laws
- Begin Chapter 5 – Conservation Laws

D. The Reynolds Transport Theorem (RTT) (Section 4-6)

1. Introduction and derivation

The Reynolds Transport Theorem (RTT) (Section 4-6)

Recall from Thermodynamics:

- A ***system*** [also called a ***closed system***] is a quantity of matter of fixed identity. *No mass can cross a system boundary*.
- A ***control volume*** [also called an ***open system***] is a region in space chosen for study. *Mass **can** cross a control surface* (the surface of the control volume).

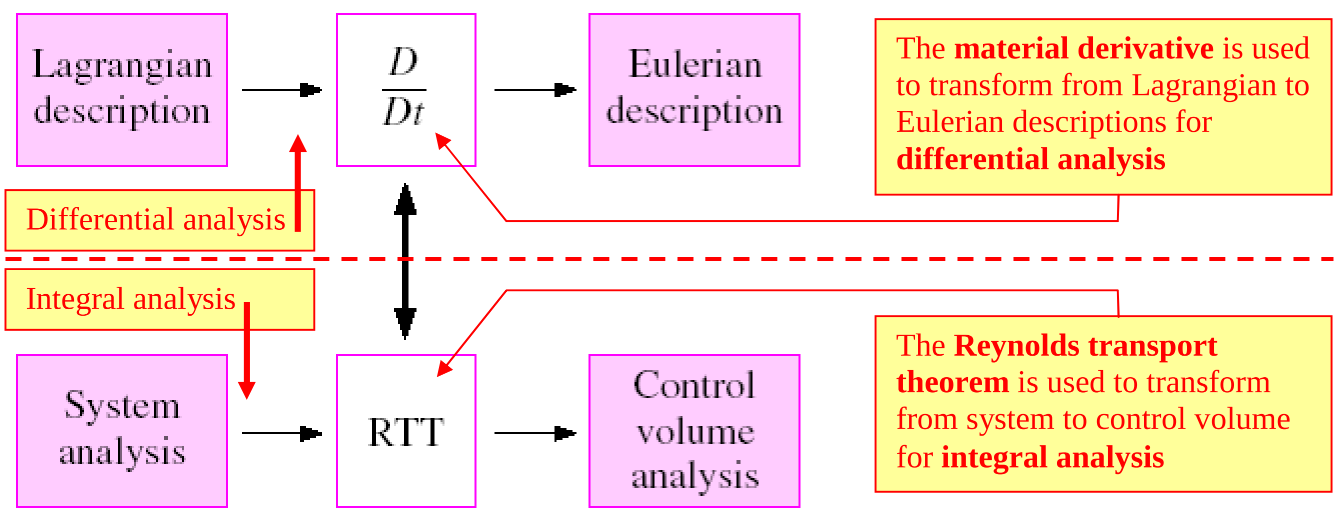
Problem

- The fundamental conservation laws (conservation of mass, energy, and momentum) *apply directly to systems*.
- However, in most fluid mechanics problems, ***control volume analysis is preferred over system analysis*** (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).

Goal of RTT

- Therefore, we need to *transform the conservation laws from a system to a control volume*. This is accomplished with the ***Reynolds transport theorem (RTT)***.

There is a direct **analogy** between the transformation from Lagrangian to Eulerian descriptions (for differential analysis using infinitesimally small fluid elements) and the transformation from systems to control volumes (for integral analysis using large, finite flow fields): t_1 • → t_2 •



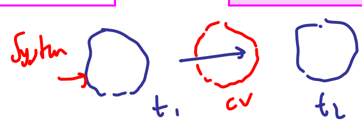


FIGURE 4-64

In both cases, the fundamental laws of physics (conservation laws) are known in the analysis on the left (Lagrangian or system), and must be transformed so as to be useful in the analysis on the right (Eulerian or control volume).

Another way to think about the RTT is that it is a *link* between the system approach and the control volume approach:

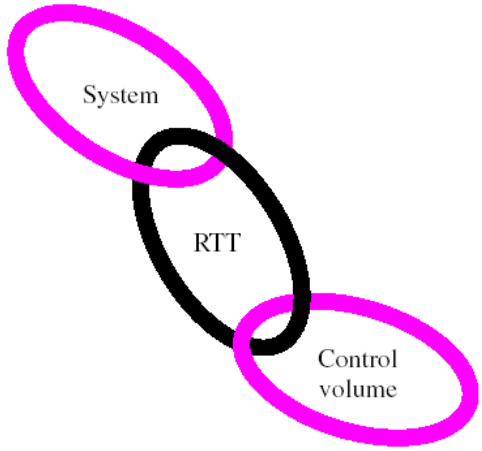


FIGURE 4-54

☆

See text for detailed derivation of the RTT. Here are some highlights:

- Let B represent any extensive property (like mass, energy, or momentum).
- Let b be the corresponding intensive property, i.e., $\boxed{b = B/m}$ (property B per unit mass).
- Our goal is to find a relationship between B_{sys} or b_{sys} (property of the system, for which we know the conservation laws) and B_{CV} or b_{CV} (property of the control volume, which we prefer to use in our analysis).
- The results are shown below in various forms:

For **fixed** (non-moving and non-deforming) control volumes,

CV = control volume
CS = control surface

RTT, fixed CV:

$$\frac{dB_{sys}}{dt} = \underbrace{\frac{d}{dt}\int_{CV} \rho b \, dV}_{\mathrm{I}} + \underbrace{\int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA}_{\mathrm{II}} \qquad \textbf{(4–41)}$$

Integrate over entire CS

Alternate RTT, fixed CV:

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA \qquad \textbf{(4–42)}$$

Since the control volume is *fixed,* the order of integration or differentiation does not matter, i.e. $\frac{d}{dt}\int_{CV}\ldots$ is the same as $\int_{CV}\frac{\partial}{\partial t}\ldots$. Thus, the two circled quantities above are *equivalent* for a fixed control volume.

Term I = rate of change of B_{sys} due to unsteadiness

Term II = rate of change of B_{sys} due to movement (convection) of the system

For **nonfixed** (moving and/or deforming) control volumes,

RTT, nonfixed CV: $$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b \, dV + \int_{\text{CS}} \rho b \vec{V}_r \cdot \vec{n} \, dA \qquad \textbf{(4–44)}$$

Note that we replace $\vec{V}$ by $\vec{V}_r$ in this version of the RTT for a moving and/or deforming control volume.

where **Error! Objects cannot be created from editing field codes.** is the **relative velocity**, i.e., the velocity of the fluid *relative to the control surface* (which may be moving or deforming),

Relative velocity: $$\vec{V}_r = \vec{V} - \vec{V}_{\text{CS}} \qquad \textbf{(4–43)}$$

We can also switch the order of the time derivative and the integral in the first term on the right, but only if we use the *absolute* (rather than the relative) velocity in the second term on the right, i.e.,

Alternate RTT, nonfixed CV: $$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n} \, dA \qquad \textbf{(4–45)}$$

Comparing Eqs. 4-45 and 4-42, we see that they are identical. Thus, the most general form of the RTT that *applies to both fixed and non-fixed control volumes* is

General RTT, nonfixed CV: $$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n} dA \qquad \textbf{(4–53)}$$

Even though this equation is most general, it is often easier *in practice* to use Eq. 4-44 for moving and/or deforming (non-fixed) control volumes because the algebra is easier.

Simplifications:

- For **steady** flow, the volume integral drops out. In terms of relative velocity,

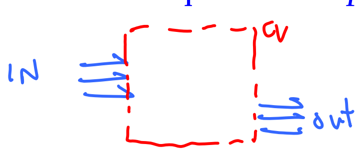
RTT, steady flow: $$\boxed{\frac{dB_{\text{sys}}}{dt} = \int_{\text{CS}} \rho b \vec{V}_r \cdot \vec{n} \, dA}$$

- For control volumes where there are **well-defined inlets and outlets**, the control surface integral can be simplified, avoiding cumbersome integrations,

Approximate RTT for well-defined inlets and outlets:

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b \, dV + \sum_{\text{out}} \underbrace{\rho_{\text{avg}} b_{\text{avg}} V_{r,\,\text{avg}} A}_{\text{for each outlet}} - \sum_{\text{in}} \underbrace{\rho_{\text{avg}} b_{\text{avg}} V_{r,\text{avg}} A}_{\text{for each inlet}} \qquad \textbf{(4–48)}$$

Note that the above equation is *approximate*, and may not always be accurate.



2. Applications of the RTT

a. Cons. of mass

$$\boxed{\frac{dm_{sys}}{dt} = 0} \; ✱$$

System form

Use RTT to convert this into CV form

For a fixed CV

Recall RTT $\quad \boxed{\frac{dB_{sys}}{dt}} = \frac{d}{dt}\int_{CV} \rho b \, d\forall + \int_{CS} \rho b (\vec{V}\cdot\vec{n})\, dA$

Let $B_{sys} = m_{sys}$, $\quad B = m = \text{mass}$

$b = B/m = 1$

$$\boxed{\frac{dm_{sys}}{dt} = 0} = \boxed{0 = \frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{CS} \rho (\vec{V}\cdot\vec{n})\, dA}$$

Eq. for a system

Eq. for a CV.

✱ Eq. of cons. of mass for a fixed CV.

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b. Cons. of Energy

For a system, 1st law of thermo.

$$\boxed{\frac{dE_{sys}}{dt} = \dot{Q}_{net\,in} + \dot{W}_{net\,in}} \rightarrow \text{Energy cons. eq. for a system}$$

Apply RTT: Let $B = E$ (total energy), $b = \frac{E}{m} = e$

(specific energy)

RTT becomes

cons. of energy for a system

$$\boxed{\frac{dE_{sys}}{dt} = \dot{Q}_{net\,in} + \dot{W}_{net\,in}} \quad \boxed{\dot{Q}_{net\,in} + \dot{W}_{net\,in} = \frac{d}{dt}\int_{CV} \rho e \, d\forall + \int_{CS} \rho e (\vec{V}\cdot\vec{n})\, dA}$$

cons. of energy for a CV

C. Cons. of Linear momentum

$$\boxed{\frac{d(m\vec{V})_{sys}}{dt} = \sum \vec{F}}$$

eq. of cons. of linear mom. for a system

Use Newton's 2nd law applies directly to a system

$$\frac{d(m\vec{V})_{sys}}{dt} = m_{sys}\frac{d\vec{V}_{sys}}{dt} + \vec{V}_{sys}\cancel{\frac{dm_{sys}}{dt}}^{\,0} = m_{sys}\vec{a}_{sys}$$

$\vec{a}_{sys}$

$$\sum \vec{F} = m_{sys}\,\vec{a}_{sys}$$

Use RTT to obtain this eq. for a CV.

Let $B = m\vec{V}$ = linear momentum

$b = \frac{B}{m} = \vec{V}$ = velocity

RTT becomes

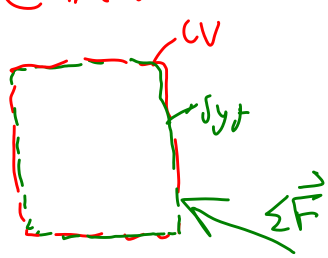
$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, d\forall + \int_{CS} \rho b (\vec{V}\cdot\vec{n})\, dA$$

$$\frac{d}{dt}(m\vec{V})_{sys} = \sum\vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, d\forall + \int_{CS} \rho\vec{V}(\vec{V}\cdot\vec{n})\, dA \quad ☆$$

linear mom. eq. for a system

linear mom. eq. for a C.V.

@ time t



Comment: @ any time t under consideration, we choose the system to be all the fluid inside the CV

END OF QUIZ 2 MATERIAL

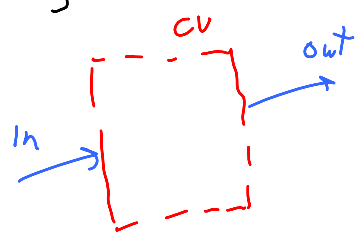
CHAPTER 5

IV CONSERVATION LAWS & THE CONTROL VOLUME (INTEGRAL) TECHNIQUE

A. Introduction

1. Overview of the techniques for solving fluid flow problems

   a. Control volume analysis [Ch. 5 & 6]



- Solve integral eqs for a CV
- Calculate gross properties only (total power output)
- Don't care (& can't compute) the details inside (treat the CV as a "black box")
- Calculations involve the CS.

   b. Dimensional Analysis & Experiment (Ch. 7 & 8)

- Don't attempt to solve any equations
- Use wind tunnels, water tunnels, models
- The dimensions are critical for scaling up model tests

   c. Differential Analysis (Ch. 9 & 10 & 15)

analytical CFD

- Solve the differential equations of motion in our flow field
- Solve for all the gory details of the flow
- Need to know boundary conditions & all geometry details

B. Conservation of Mass (Sec. 5-2)

1. Equations & Definitions

- From RTT (see notes above) for a fixed CV,

$$\boxed{\underbrace{\frac{d}{dt}\int_{CV} \rho \, d\forall}_{\mathrm{I}} + \underbrace{\int_{CS} \rho (\vec{V}\cdot\vec{n})\, dA}_{\mathrm{II}} = 0} \quad (1)$$

In words:

rate of change of mass within the CV	+	net rate of mass flow **out** of the CV through the CS	=	0

Comment about signs:

Since there are only 2 terms in this eq., & the RHS = 0,
(right hand side)

- If term I is ⊕ve, term II must be ⊖ve.
- If term I is ⊖ve, term II must be ⊕ve.
- If term I is 0, term II must also be 0.

So, for steady flow (term I = 0), Eq (1) reduces to

$$\boxed{\int_{CS} \rho (\vec{V}\cdot\vec{n})\, dA = 0 \quad \text{(for steady flow)}} \quad ☆$$