**Today, we will**:

- Derive and discuss the linear momentum equation for a control volume (Chapter 6)
- Discuss the momentum flux correction factor, $\beta$
- Discuss all the various forces acting on a control volume
- Do some example problems – Linear momentum equation for a control volume

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**E. The Linear Momentum Equation for a Control Volume (Chapter 6)**

1. Equations and Definitions

**Recall from the RTT at the end of Chapter 4,** → recall we set $B = m\vec V$, $b = \vec V$

$$\sum \vec F = \frac{d}{dt}\int_{CV} \rho \vec V dV + \int_{CS} \rho \vec V \left(\vec V \cdot \vec n\right) dA$$

For a fixed CV

Total force (***vector***) acting on the control volume

Rate of change of linear momentum inside the control volume

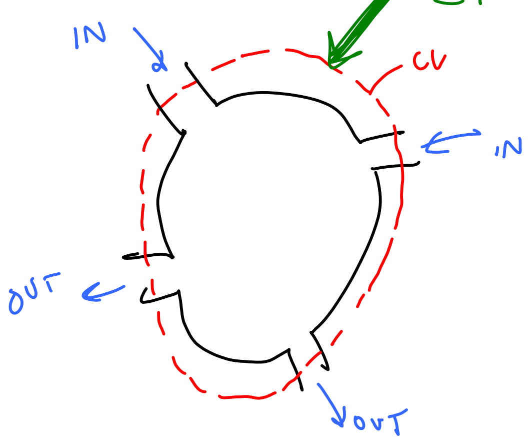
Net rate of linear momentum flow out of the control volume

Use *relative* velocity $\vec V_r$ here if we have a moving or deforming control volume

This is a vector eq. → need 3 components in general

Simplification for a CV with well-defined inlets & outlets

$\sum \vec F$ = net force acting on the CV




The CS integral term becomes

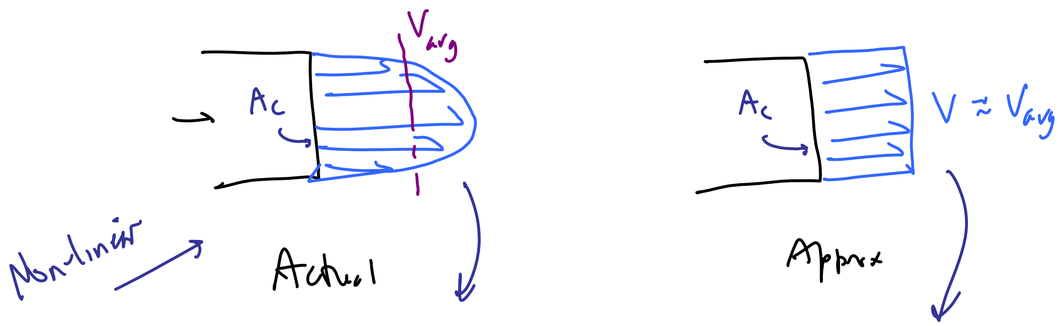
$$\int_{CS} \rho \vec V \left(\vec V \cdot \vec n\right) dA \approx \sum_{out} \dot m \vec V_{avg} - \sum_{in} \dot m \vec V_{avg}$$

drop the subscript "avg", so,

$$\boxed{\sum \vec F \approx \frac{d}{dt}\int_{CV} \rho \vec V d\forall + \sum_{out} \dot m \vec V - \sum_{in} \dot m \vec V} \qquad (1)$$

2. The momentum flux correction factor, $\beta$

A real inlet or outlet does not have uniform velocity




$$\underline{\rho \int_{A_c} V^2 \, dA} \quad > \quad \underline{\rho V_{avg} A_c V_{avg}}$$

Introduce
$$\boxed{\beta \equiv \frac{1}{A_c} \int_{A_c} \left( \frac{V}{V_{avg}} \right)^2 dA_c}$$

For any real velocity profile, $\beta > 1$

e.g.'s of $\beta$:

- Uniform flow $\rightarrow \boxed{\beta = 1}$
- fully developed laminar pipe flow $- \boxed{\beta = \frac{4}{3}}$
- " " turbulent " " $- \beta \simeq 1.01 - 1.04$

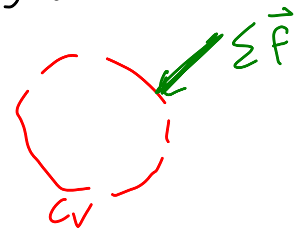
let's use $\boxed{\beta \simeq 1.02}$

Eq. (1) becomes

$$\boxed{\sum \vec{F} = \frac{d}{dt} \int_{CV} \rho \vec{V} \, d\forall + \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V}} \quad (2)$$

More useful eq for engineering analysis

3. Forces acting on a CV




$$\sum \vec{F} = \underbrace{\sum \vec{F}_{body}}_{\substack{\text{weight of} \\ \text{the CV} \\ \text{(gravity)}}} + \underbrace{\sum \vec{F}_{surface}}$$

- pressure
- viscous
- other (e.g. tension in a cable, force on a bolt, etc)

$$\boxed{\sum \vec{F} = \sum \vec{F}_{gravity} + \sum \vec{F}_{pressure} + \sum \vec{F}_{viscous} + \sum \vec{F}_{other} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, d\forall + \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V}} \quad (3)$$

Most useful form of the linear momentum CV eq.

Comment: • In most problems, the unsteady term is zero (SSSF)

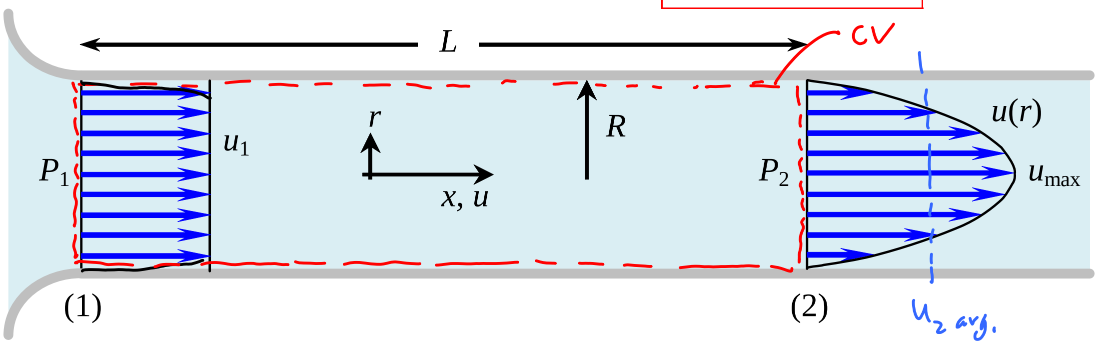
→ • The viscous term is often either zero or our unknown

4. Examples

**Example: Friction force in a pipe**

**Given**: Consider steady, laminar, incompressible, axisymmetric flow of a liquid in a pipe as sketched. At the inlet (1) there is a nice bell mouth, and the velocity is nearly uniform (except for a very thin boundary layer, not shown).

- At (1), $u = u_1$ = constant, $v = 0$, and $w = 0$. $P_1$ is measured.
- At (2), the flow is fully developed and parabolic: $\boxed{u_2 = u_{max}\left(1 - \frac{r^2}{R^2}\right)}$. $P_2$ is measured.




**To do**: Calculate the total friction force acting on the fluid by the pipe wall from 1 to 2.

**Solution**:

- First step: Draw a CV

· Cons. of mass

$$\dot{m}_{in} = \dot{m}_{out}$$

$$\underbrace{\cancel{\rho}}_{\rho = const} u_1 \underbrace{\cancel{A_1} = \cancel{\rho} u_{2\,avg} \cancel{A_2}}_{const\ (A_1 = A_2)} \Rightarrow \boxed{u_{2\,avg} = u_1}$$

· Momentum flux correction factor

@ (1) $\beta_1 = 1$

@ (2) $\beta_2 = \frac{4}{3}$

(fully dev. lam. pipe flow)

• Now use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \cancel{\sum \vec{F}_{\text{gravity}}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \cancel{\sum \vec{F}_{\text{other}}} = \cancel{\frac{d}{dt}\int_{CV} \rho \vec{V} dV} + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

no weight in x

$P_1 A_1 - P_2 A_2$

$-F_{\text{friction, wall on liquid}}$

[our unknown]

none

Steady

1 inlet, 1 outlet

$\beta_2 \dot{m} u_{2_{avg}} - \beta_1 \dot{m} u_1$

$\beta_2 \dot{m} u_1 - \beta_1 \dot{m} u_1$

In x-direction

Solve for the unknown:

$$F_{\text{friction, wall on liquid}} = (P_1 - P_2)A + \beta_1 \dot{m} u_1 - \beta_2 \dot{m} u_1$$

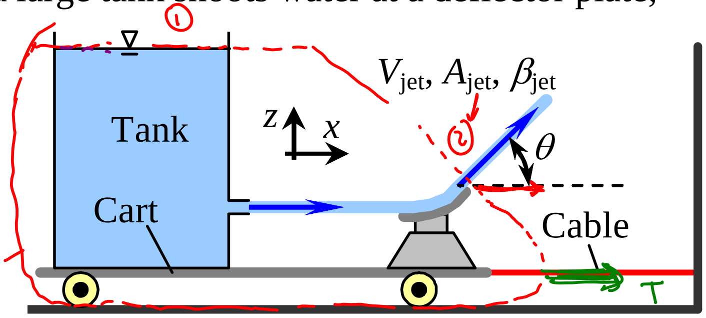
$A = \pi R^2$, $\beta_1 = 1$, $\beta_2 = 4/3$

$$\dot{m} = \rho u_{avg} A = \rho u_1 \pi R^2$$

$$\boxed{F_{\text{friction, wall on liquid}} = \pi R^2 \left[P_1 - P_2 - \frac{1}{3}\rho u_1^2\right]}$$

## Example: Tension in a cable

**Given**: A cart with frictionless wheels and a large tank shoots water at a deflector plate, turning it by angle $\theta$ as sketched. The cart tries to move to the left, but a cable prevents it from doing so. At the exit of the deflector, the water jet area $A_{\text{jet}}$, its average velocity $V_{\text{jet}}$, and its momentum flux correction factor $\beta_{\text{jet}}$ are known.




**To do**: Calculate the tension $T$ in the cable.

**Solution**:

• First step:

• Second step: Use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

IN x-direction

$$\sum F_x = \cancel{\sum F_{x,\text{grav}}} + \cancel{\sum F_{x,\text{press}}} + \cancel{\sum F_{x,\text{visc}}} + \sum F_{x,\text{other}} = \sum_{\text{out}} \beta \dot{m} u_{\text{avg}} - \cancel{\sum_{\text{in}} \beta \dot{m} u_{\text{avg}}}$$

(0 in x)

0 ($P = P_{atm}$ everywhere)

0 (none)

$\parallel$ $T$

$\beta_{jet}$

$\rho V_{jet} A_{jet}$

$V_{jet} \cos\theta$

$V_1 \approx 0$ @ inlet

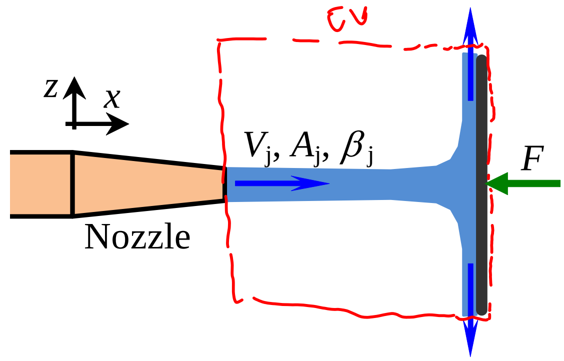
Solve for T:

$$\boxed{T = \beta_{jet} \rho V_{jet}^2 A_{jet} \cos\theta}$$

**Example: Force imparted by a water jet hitting a stationary plate**

**Given**: A horizontal water jet of area $A_j$, average velocity $V_j$, and momentum flux correction factor $\beta_j$ impinges normal to a stationary vertical flat plate.

**To do**: Calculate the horizontal force $F$ required to keep the plate from moving.




**Solution**:

- First step: Draw CV
- Second step: Use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \cancel{\frac{d}{dt}\int_{CV} \rho \vec{V} dV} + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

steady

$$\sum F_x = \cancel{\sum F_{x,grav}} + \cancel{\sum F_{x,press}} + \cancel{\sum F_{x,visc}} + \sum F_{x,other} = \cancel{\beta \dot{m} u} - \beta \dot{m} u$$

0 in x ; $P = P_{atm}$ everywhere ; Wise CV ; $\sum F_{x,other} = -F$ ; (outlet) $u = 0$ @ outlet ; (inlet)

Solve for F

$$\boxed{F = \beta_j \rho V_j^2 A_j}$$