Today, we will:

- Do some more example problems – linear CV momentum equation
- Discuss the control volume equation for angular momentum

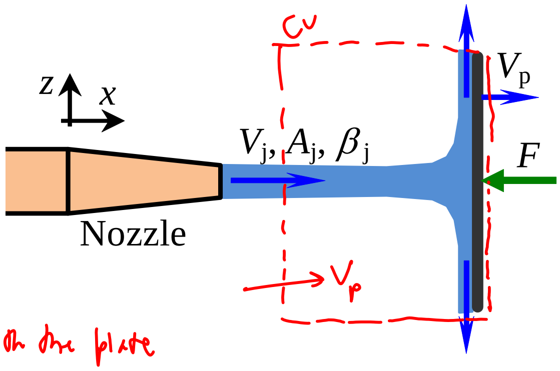
E. The Linear Momentum Equation for a Control Volume (continued)

4. Examples (continued)

Example: Force imparted by a water jet hitting a *moving* plate

Given: A horizontal water jet of area A_j, average velocity V_j, and momentum flux correction factor β_j impinges normal to a *moving* vertical flat plate. The plate moves to the right at speed V_p.

To do: Calculate the horizontal force F required to keep the plate moving at constant speed V_p.



Solution:

- First step: - Pick a CV → use a moving CV, moving with the plate
- Second step: Use the approximate, most useful form of the linear momentum equation, in the x-direction, for a moving CV, but *steady*:

$$\boxed{\sum F_x = \sum F_{x,\text{ gravity}} + \sum F_{x,\text{ pressure}} + \sum F_{x,\text{ viscous}} + \sum F_{x,\text{ other}} = \sum_{\text{out}} \beta \dot{m} u_r - \sum_{\text{in}} \beta \dot{m} u_r}$$

$u_r = u_{\text{absolute}} - u_{cs}$

This is an inertial reference frame since it is not accelerating.

↓

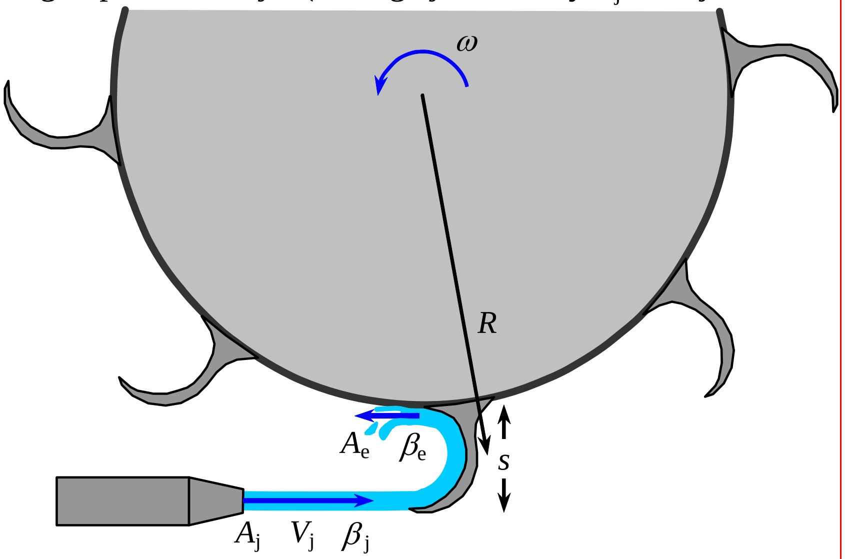
Replace V_j by $V_j - V_p$ in our eq.

Last lecture, we had $F = \beta_j \rho V_j^2 A_j$ for a fixed plate

Here, $\boxed{F = \beta_j \rho (V_j - V_p)^2 A_j}$

Example: Force on a bucket of a Pelton-type (impulse) hydroturbine

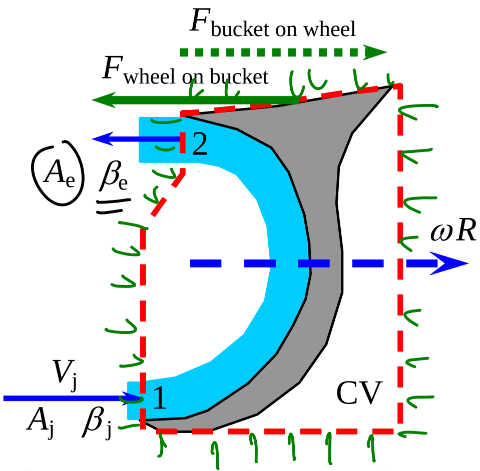
Given: An impulse turbine is driven by a high-speed water jet (average jet velocity V_j over jet area A_j, with momentum flux correction factor β_j) that impinges on turning buckets attached to a turbine wheel as shown. The turbine wheel rotates at angular velocity ω, and is horizontal; therefore, gravity effects are not important in this problem. (The view in the sketch is from the top.) The turning buckets turn the water approximately 180 degrees, and the water exits the bucket over exit cross-sectional area A_e with exit momentum flux correction factor β_e. For simplicity, we approximate that the bucket dimension s is much smaller than turbine wheel radius R ($s << R$).



(**a**) **To do**: Calculate the force of the bucket on the turbine wheel, $F_{\text{bucket on wheel}}$, at the instant in time when the bucket is in the position shown.

(**b**) **To do**: Calculate the power delivered to the turbine wheel.

Solution: We choose a control volume surrounding the bucket, cutting through the water jet at the inlet to the bucket, and cutting through the water exiting the bucket. Note that this is a *moving control volume*, moving to the right at speed ωR. We also cut through the welded joint between the bucket and the turbine wheel, where the force $F_{\text{bucket on wheel}}$ is to be calculated. Because of Newton's third law, the force acting *on the control volume* at this location is equal in magnitude, but opposite in direction, and we call it F~~bucket on wheel~~. wheel on bucket



Since the pressure through an incompressible jet exposed to atmospheric air is equal to P_{atm}, the pressure at the inlet (1) is equal to P_{atm}, and the pressure at the exit (2) is also equal to P_{atm}.

Solution to be completed in class.

$\vec{V}_r = \vec{V} - \vec{V}_{cs}$ at all inlets & outlets

@ ① (inlet) $\vec{V}_r = V_j \vec{i} - \omega R \vec{i}$ → In x-direction: $\boxed{u_{r_1} = V_j - \omega R}$

· Cons. of mass (Quasi steady) → @ this instant in time we are approx. the flow as "steady"

$$\sum \dot{m}_{in} = \sum \dot{m}_{out} \rightarrow \cancel{\rho} V_{r_{in}} A_{in} \overset{①}{} = \cancel{\rho} V_{r_{out}} A_{out} \overset{②}{}$$

$$(V_j - \omega R) A_j = V_{r_2} A_e \rightarrow \boxed{V_{r_2} = (V_j - \omega R)\frac{A_j}{A_e}}$$

☆ $\boxed{u_{r_2} = -(V_j - \omega R) A_j / A_e}$

- Use the *x*-component of the steady linear momentum equation for a moving CV,

$$\sum F_x = \cancel{\sum F_{x,\,\text{gravity}}} + \cancel{\sum F_{x,\,\text{pressure}}} + \cancel{\sum F_{x,\,\text{viscous}}} + \sum F_{x,\,\text{other}} = \sum_{\text{out}} \beta \dot{m} u_r - \sum_{\text{in}} \beta \dot{m} u_r$$

0 in x

($P_1 = P_2$)

(wise CV)

$-F_{\text{bucket on wheel}}$

⊕ ve

$\beta_e \dot{m} u_{r_2} - \beta_j \dot{m} u_{r_1}$

- We need $\dot{m}$ → evaluate at the inlet or outlet (relative to the moving CV)

$$\boxed{\dot{m} = \rho (V_j - \omega R) A_j}$$

· Solve for

$$\boxed{F_{\text{bucket on wheel}} = \rho (V_j - \omega R)^2 A_j \left(\beta_j + \beta_e \frac{A_j}{A_e}\right)} \ \star$$

(b) $\dot{W}_{\text{wheel}}$ = Torque × ang. velocity = $T \cdot \omega$

$= F_{\text{bucket on wheel}} \cdot R \cdot \omega$

$$\boxed{\dot{W}_{\text{wheel}} = \rho (V_j - \omega R)^2 A_j \left(\beta_j + \beta_e \frac{A_j}{A_e}\right) R\,\omega}$$

(for one bucket)

F. Cons. of Angular Momentum

1. Eqs & Definitions

· Use RTT → use $B = \vec{H}$ = angular momentum = $\vec{r} \times m\vec{V}$

cross product

$b = \frac{B}{m}$, $b = \vec{r} \times \vec{V}$ (b is a vector)

Angular Momentum Control Volume Analysis
(Section 6-6, Çengel and Cimbala)

1. Equations and definitions

See the derivation in the book, using the Reynolds transport theorem. The result is:

General:
$$\sum \vec{M} = \frac{d}{dt}\int_{\text{CV}} (\vec{r}\times\vec{V})\rho\, dV + \int_{\text{CS}} (\vec{r}\times\vec{V})\rho(\boxed{\vec{V}_r}\cdot\vec{n})\, dA \qquad \textbf{(6–47)}$$

(Relative velocity)

which can be stated as

$$\left(\begin{array}{c}\text{The sum of all}\\ \text{external moments}\\ \text{acting on a CV}\end{array}\right) = \left(\begin{array}{c}\text{The time rate of change}\\ \text{of the angular momentum}\\ \text{of the contents of the CV}\end{array}\right) + \left(\begin{array}{c}\text{The net flow rate of}\\ \text{angular momentum}\\ \text{out of the control}\\ \text{surface by mass flow}\end{array}\right)$$

We simplify the control surface integral for cases in which there are well-defined inlets and outlets, just as we did previously for mass, energy, and momentum. The result is:

$$\sum \vec{M} = \frac{d}{dt}\int_{\text{CV}} (\vec{r}\times\vec{V})\rho\, dV + \sum_{\text{out}} \vec{r}\times\dot{m}\vec{V} - \sum_{\text{in}} \vec{r}\times\dot{m}\vec{V} \qquad \textbf{(6–50)}$$

Note that we cannot define an “angular momentum flux correction factor” like we did previously for the kinetic energy and momentum flux terms. Furthermore, many problems we consider in this course are *steady*. For steady flow, Eq. 6-50 reduces to:

Steady flow:
$$\sum \vec{M} = \sum_{\text{out}} \vec{r}\times\dot{m}\vec{V} - \sum_{\text{in}} \vec{r}\times\dot{m}\vec{V} \qquad \textbf{(6–51)}$$

$$\begin{array}{c}\text{Net moment or}\\ \text{torque acting on}\\ \text{the control volume}\\ \text{by external means}\end{array} = \begin{array}{c}\text{Rate of flow of}\\ \text{angular momentum}\\ \textit{out}\text{ of the control}\\ \text{volume by mass flow}\end{array} - \begin{array}{c}\text{Rate of flow of}\\ \text{angular momentum}\\ \textit{into}\text{ the control}\\ \text{volume by mass flow}\end{array}$$

Finally, in many cases, we are concerned about only *one* axis of rotation, and we simplify Eq. 6-51 to a scalar equation,

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V \qquad \textbf{(6–52)}$$

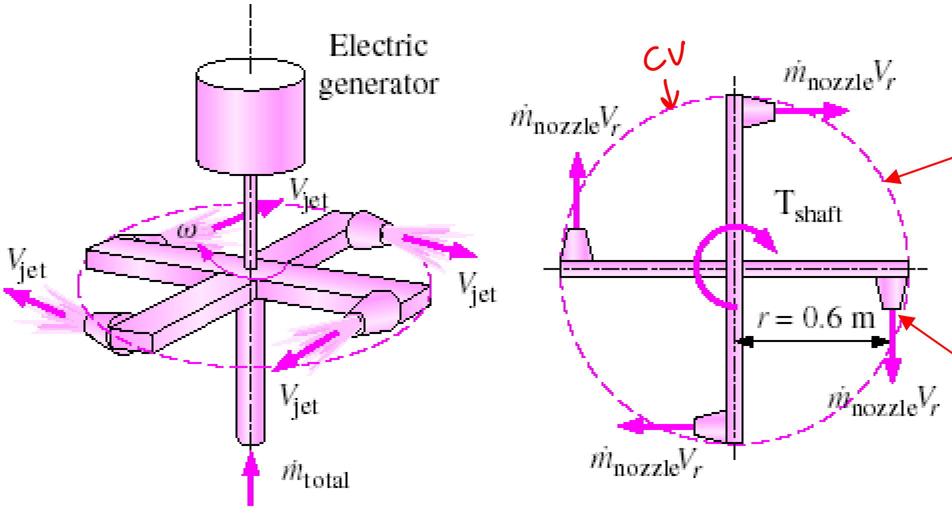
Equation 6-52 is the form of the angular momentum control volume equation that we will most often use, noting that *r* is the shortest distance (i.e. the *normal* distance) between the point about which moments are taken and the *line of action* of the force or velocity being considered. By convention, *counterclockwise moments are positive*.

2. Examples

See Examples 6-8 and 6-9 in the book. Example 6-9 is discussed in more detail here.

EXAMPLE 6–9 Power Generation from a Sprinkler System

A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head, as shown in Fig. 6–38. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.



We choose the stationary control volume shown

$P = P_{atm}$ at the nozzle exits

FIGURE 6-37

Assumptions **1** The flow is cyclically steady (i.e., steady from a frame of reference rotating with the sprinkler head). **2** The water is discharged to the atmosphere, and thus the gage pressure at the nozzle exit is zero. **3** Generator losses and air drag of rotating components are neglected. **4** The nozzle diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

Properties We take the density of water to be 1000 kg/m³ = 1 kg/L.

Analysis We take the disk that encloses the sprinkler arms as the control volume, which is a stationary control volume.

Conservation of mass:

The conservation of mass equation for this steady-flow system is $\dot m_1 = \dot m_2 = \dot m_{total}$. Noting that the four nozzles are identical, we have $\dot m_{nozzle} = \dot m_{total}/4$ or $\dot V_{nozzle} = \dot V_{total}/4$ since the density of water is constant. The average jet exit velocity relative to the nozzle is

$$V_{jet} = \frac{\dot V_{nozzle}}{A_{jet}} = \frac{\overset{20/4}{5\ \text{L/s}}}{\underline{[\pi(0.01\ \text{m})^2/4]}}\left(\frac{1\ \text{m}^3}{1000\ \text{L}}\right) = \underline{63.66\ \text{m/s}}$$

The angular and tangential velocities of the nozzles are

$$\omega = 2\pi\dot{n} = 2\pi(300\text{ rev/min})\left(\frac{1\text{ min}}{60\text{ s}}\right) = 31.42\text{ rad/s}$$

$$V_{\text{nozzle}} = r\omega = (0.6\text{ m})(31.42\text{ rad/s}) = 18.85\text{ m/s}$$

That is, the water in the nozzle is also moving at a velocity of 18.85 m/s in the opposite direction when it is discharged. Then the average velocity of the water jet relative to the control volume (or relative to a fixed location on earth) becomes

$$V_r = V_{\text{jet}} - V_{\text{nozzle}} = 63.66 - 18.85 = 44.81\text{ m/s}$$

Note: We must use the velocity relative to the control volume, which in this case is the *absolute* velocity, since our control volume is fixed (not moving).

Conservation of angular momentum:

Noting that this is a cyclically steady-flow problem, and all forces and momentum flows are in the same plane, the angular momentum equation can be approximated as $\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$, where r is the moment arm, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

The free-body diagram of the disk that contains the sprinkler arms is given in Fig. 6–38. Note that the moments of all forces and momentum flows passing through the axis of rotation are zero. The momentum flows via the water jets leaving the nozzles yield a moment in the clockwise direction and the effect of the generator on the control volume is a moment also in the clockwise direction (thus both are negative). Then the angular momentum equation about the axis of rotation becomes

$$-\text{T}_{\text{shaft}} = -4r\dot{m}_{\text{nozzle}}V_r \qquad \text{or} \qquad \boxed{\text{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_r}$$

Be careful with the *signs.*

Substituting, the torque transmitted through the shaft is determined to be

$$\text{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_r = (0.6\text{ m})(20\text{ kg/s})(44.81\text{ m/s})\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right) = 537.7\text{ N}\cdot\text{m}$$

since $\dot{m}_{\text{total}} = \rho\dot{V}_{\text{total}} = (1\text{ kg/L})(20\text{ L/s}) = 20\text{ kg/s}$.

Calculation of the shaft power:

Then the ~~power generated~~ Shaft power becomes

$$\dot{W} = \overset{\omega}{\boxed{2\pi\dot{n}}}\,T_{\text{shaft}} = \omega T_{\text{shaft}} = (31.42\ \text{rad/s})(537.7\ \text{N}\cdot\text{m})\left(\frac{1\ \text{kW}}{1000\ \text{N}\cdot\text{m/s}}\right) = \underline{16.9\ \text{kW}}$$

Therefore, this sprinkler-type turbine has the potential to produce 16.9 kW of power.

The actual power generated will be less than this because of generator inefficiencies. We can calculate the generated power as

$$\boxed{\dot{W}_{\text{electric}} = \eta_{\text{generator}}\dot{W}_{\text{shaft}}}$$

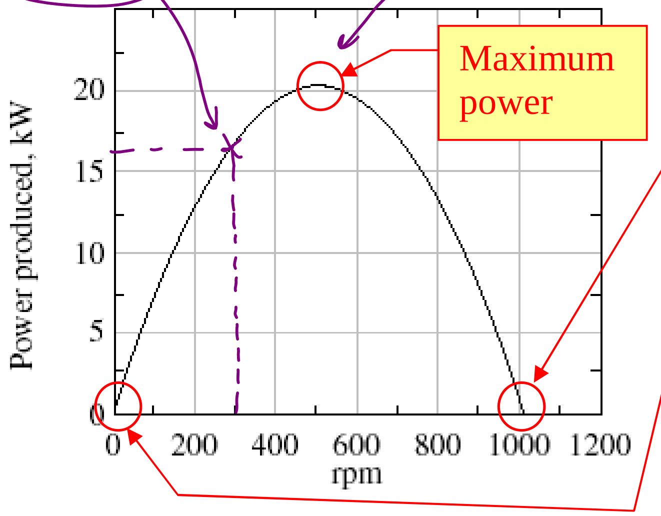
Discussion To put the result obtained in perspective, we consider two limiting cases. In the first limiting case, the sprinkler is stuck and thus the angular velocity is zero. The torque developed will be maximum in this case since $V_{\text{nozzle}} = 0$ and thus $V_r = V_{\text{jet}} = 63.66$ m/s, giving $T_{\text{shaft, max}} = 764\ \text{N}\cdot\text{m}$. But the power generated will be zero since the shaft does not rotate.

In the second limiting case, the shaft is disconnected from the generator (and thus both the torque and power generation are zero) and the shaft accelerates until it reaches an equilibrium velocity. Setting $T_{\text{shaft}} = 0$ in the angular momentum equation gives $V_r = 0$ and thus $V_{\text{jet}} = V_{\text{nozzle}} = 63.66$ m/s. The corresponding angular speed of the sprinkler is

$$\dot{n} = \frac{\omega}{2\pi} = \frac{V_{\text{nozzle}}}{2\pi r} = \frac{63.66\ \text{m/s}}{2\pi(0.6\ \text{m})}\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 1013\ \text{rpm}$$

Our operating point

BEST EFFICIENCY POINT → Best case



Maximum power

Limiting case: when ω = maximum, the shaft power is zero because there is motion, but *no torque*.

Limiting case: when $\omega = 0$, the shaft power is zero because there is torque, but *no motion*.

FIGURE 6-38

The variation of power produced with angular speed.

V. Dimensional Analysis & Similarity (CH. 7)

A. Primary Dimensions (there are 7) or fundamental dimensions

dimen.	symbol
mass	m
length	L
time	t
temperature	T
Electric current	I
Amt of light	C
Amt of matter	N

All other dimensions can be formed by a combination of these 7.

Eg. Force:

$$\{Force\} = \{ma\} = \boxed{\left\{ m \frac{L}{t^2} \right\} = \{F\}}$$