Today, we will:

- Discuss dimensional homogeneity
- Discuss dimensional analysis and similarity, and the method of repeating variables

IV. DIMENSIONAL ANALYSIS AND MODELING (Chapter 7)

A. Primary Dimensions (see previous lecture): {m}, {L}, {t}, {T}, {I}, {C}, {N}.
All other dimensions can be formed by combination of these 7 primary dimensions.

Example: Primary dimensions – shear stress, force per unit length, and power

(*a*) **Given**: In fluid mechanics, shear stress τ is expressed in units of N/m^2.

To do: Express the primary dimensions of τ, i.e., write an expression for $\{\tau\}$.

Solution:

$$\{\tau\} = \left\{\frac{\text{force}}{\text{area}}\right\} = \left\{\frac{mL}{t^2 L^2}\right\} = \boxed{\left\{\frac{m}{L t^2}\right\}} \quad \text{or} \quad \boxed{\{m^1 L^{-1} t^{-2}\}}$$

(*b*) **Given**: Ray is conducting an experiment in which quantity a has dimensions of force per unit length.

To do: Express the primary dimensions of a, i.e., write an expression for $\{a\}$.

Solution:

$$\{a\} = \left\{\frac{F}{L}\right\} = \left\{\frac{m\cancel{L}}{t^2\cancel{L}}\right\} = \boxed{\left\{\frac{m}{t^2}\right\}}$$

(*c*) **Given**: Power $\dot{W}$ has the dimensions of energy per unit time.

To do: Write the dimensions of power in terms of primary dimensions.

Solution:

$$\{\text{Power}\} = \left\{\frac{\text{energy}}{\text{time}}\right\} = \left\{\frac{\text{Force}\cdot L}{t}\right\} = \left\{\frac{mL}{t^2}\frac{L}{t}\right\} = \boxed{\left\{\frac{mL^2}{t^3}\right\} = \{\dot{W}\}}$$

B. Dimensional Homogeneity

☆ All additive terms in an equation must have the same dimensions

e.g. Momentum CV eq.

$$\sum \vec{F} = \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V}$$

$$\left\{\frac{mL}{t^2}\right\} \qquad \left\{1 \cdot \frac{m}{t} \cdot \frac{L}{t}\right\} \qquad \left\{\frac{mL}{t^2}\right\}$$

$$\left\{1 \cdot \frac{m}{t} \cdot \frac{L}{t}\right\} = \left\{\frac{mL}{t^2}\right\}$$

C. Dimensional Analysis & Similarity

1. Purposes of dim. anal.

- To help plan & carry out experiments
- To help obtain scaling laws → eg. test a small model, need to scale up to a prototype
- To (sometimes) predict trends & equations within a constant

~~B. Dimensional Homogeneity~~

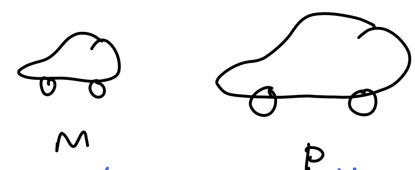
2. Similarity

Model → usually smaller (but sometimes larger)

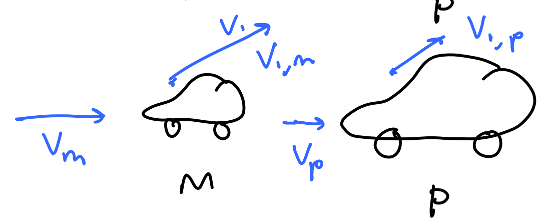
Prototype → full scale

To properly scale the model to the prototype, we need to achieve complete similarity

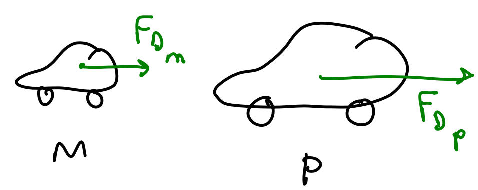
1. Geometric similarity (proportional geometry)



2. Kinematic similarity (proportional velocities)



3. Dynamic similarity (proportional forces)



☆ If we achieve dynamic or complete similarity, we can exactly scale up from model to prototype

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We write a functional relationship between several nondimensional parameters, called Π's

$\{\Pi_1\} = \{1\}$ dimensionless

$$\boxed{\underbrace{\Pi_1}_{\text{Dependent } \Pi} = \text{fnc}\left(\underbrace{\Pi_2, \Pi_3, \Pi_4 \ldots \Pi_k}_{\text{Independent } \Pi\text{s}}\right)}$$

For complete similarity between a model & a prototype,

$$\text{If} \quad \begin{aligned} \Pi_{2,m} &= \Pi_{2,p} \\ \&\ \Pi_{3,m} &= \Pi_{3,p} \\ &\vdots \\ \&\ \Pi_{k,m} &= \Pi_{k,p} \end{aligned} \qquad \text{Then} \quad \Pi_{1,m} = \Pi_{1,p}$$

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3. The Method of Repeating Variables

See Fig. 7-22 & Table 7-2

We will learn by examples

(Also see several examples in the textbook)

Steps in the Method of Repeating Variables

There are 6 steps that comprise the method of repeating variables. These are listed concisely in Fig. 7-22 in the text, as repeated below:

The Method of Repeating Variables

Step 1: List the parameters in the problem and count their total number n.

Step 2: List the primary dimensions of each of the n parameters.

Step 3: Set the *reduction* j as the number of primary dimensions. Calculate k, the expected number of Π's,

$$k = n - j$$

Step 4: Choose j *repeating parameters*.

Step 5: Construct the k Π's, and manipulate as necessary.

Step 6: Write the final functional relationship and check your algebra.

Step 4 is often the most difficult or mysterious step. There are guidelines provided in Table 7-3, but it takes practice to know which repeating variables to choose wisely.

FIGURE 7–22

A concise summary of the six steps that comprise the *method of repeating variables*.

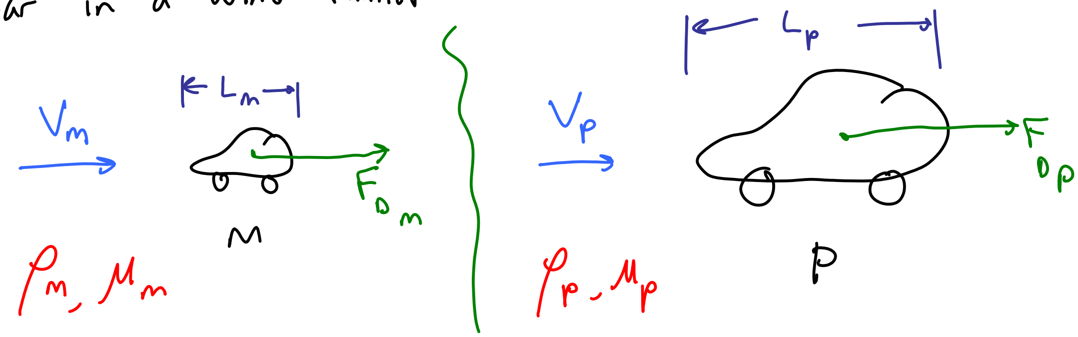
The final functional relationship is given as the *dependent* Π, Π_1, as a function of the *independent* Π's, Π_2, Π_3, … , Π_k, i.e., $\boxed{\Pi_1 = f\left(\Pi_2, \Pi_3, \ldots, \Pi_k\right)}$

Guidelines for choosing the repeating variables in Step 4 of the method of repeating variables: (See Table 7-3 in the text for more details):

1. Never pick the *dependent* variable. Otherwise, it may appear in all the Π's, which is undesirable.
2. The chosen repeating parameters must not *by themselves* be able to form a dimensionless group. Otherwise, it would be impossible to generate the rest of the Π's.
3. The chosen repeating parameters must represent *all* the primary dimensions in the problem.
4. Never pick parameters that are already dimensionless. These are Π's already, all by themselves.
5. Never pick two parameters with the *same* dimensions or with dimensions that differ by only an exponent.
6. Whenever possible, choose dimensional constants over dimensional variables so that only *one* Π contains the dimensional variable.
7. Pick common parameters since they may appear in each of the Π's.

4. Examples

Model car in a wind tunnel



Step 1 List the parameters & count them (include the dependent one)

$$F_D = \text{fnc}(V, L, \rho, \mu)$$

μ: units $= \frac{kg}{m \cdot s}$

$\boxed{n = 5}$

Step 2 List the primary dimensions of each parameter

$$\boxed{\left\{ m\frac{L}{t^2} \right\} \quad \left\{ \frac{L}{t} \right\} \quad \{L\} \quad \left\{ \frac{m}{L^3} \right\} \quad \left\{ \frac{m}{L \cdot t} \right\}}$$

Step 3 Guess the reduction, j

Usually, j = # of primary dimensions in the problem

Here, we have m, L, t → guess j = 3

[If this is wrong, we will reduce it by 1 & try again]

Buckingham Pi Theorem

$$\boxed{k = n - j}$$

k: # of Πs; n: step 1; j: step 2

→ Here, $k = n - j$
$= 5 - 3 = 2$

We expect 2 Πs

We expect to get $\boxed{\Pi_1 = \text{fnc}(\Pi_2)}$

Step 4   Choose j repeating variables

I   Pick $V, L, \text{ \& } \rho$

Step 5   Construct the Πs

$$\boxed{\Pi_1 = F_D V^a L^b \rho^c}$$

Now we solve for a, b, c

Force $\{\Pi_1\}$ to be $\{1\}$

$$\{\Pi_1\} = \{1\} = \{m^0 L^0 t^0\} = \{F_D V^a L^b \rho^c\}$$

from Step 2,

$$\{m^0 L^0 t^0\} = \{(m^1 L^1 t^{-2})(L^a t^{-a})(L^b)(m^c L^{-3c})\}$$

m: $\quad m^0 = m^1 m^c \rightarrow m^{1+c} \rightarrow 1+c=0$

L: $\quad L^0 = L^1 L^a L^b L^{-3c} \rightarrow L^{1+a+b-3c} \rightarrow 0 = 1+a+b-3c$

t: $\quad t^0 = t^{-2} t^{-a} \rightarrow 0 = -2-a$

Solve simult. for a, b, c,

$$\boxed{\begin{aligned} c &= -1 \\ a &= -2 \\ b &= -2 \end{aligned}}$$

$$\therefore \boxed{\Pi_1 = F_D V^{-2} L^{-2} \rho^{-1}}$$