***NOTE*: Lecture 27 was cancelled, Fall 2012. Review that material on your own.**

Today, we will:

- Begin our discussion of Chapters 9 and 15 – **Differential Analysis of Fluid Flow**

VII DIFFERENTIAL ANALYSIS OF FLUID FLOW

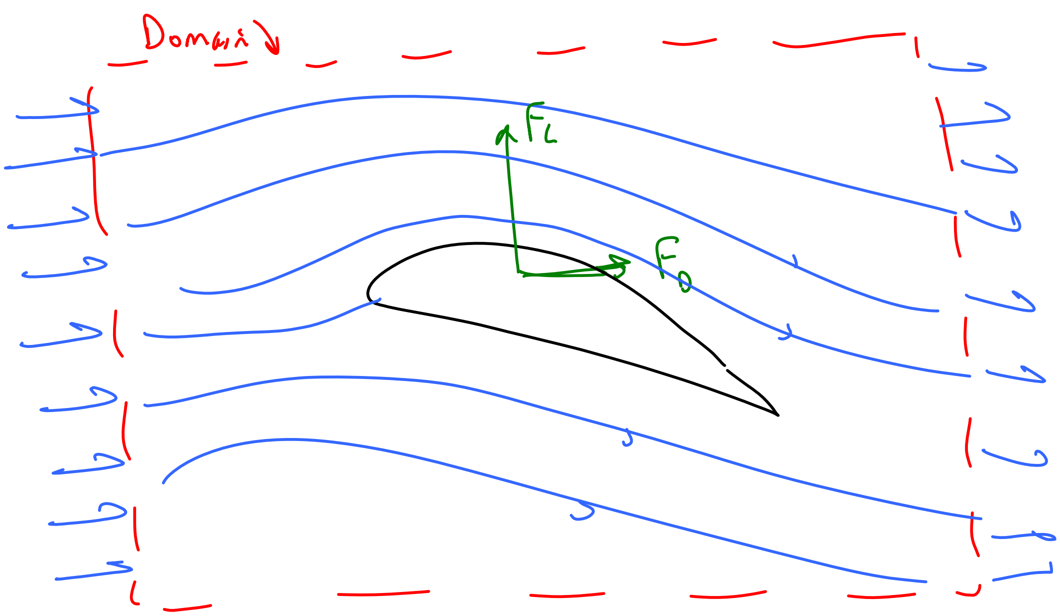
A. Introduction

So far, we have used CV technique

↳ obtain only gross properties, no details inside the CV

In fact the inside of the CV is a "black box"

Now → Do differential analysis → solve all the details inside the domain



How to solve the diff. eq's?

1) Analytically (w/ pencil & paper) [CH 9] (simple problems only)

2) Computationally (CFD) [CH 15] (complex problems)

Computational Fluid Dynamics

B. Technique of Differential Analysis (≈ same for analytical vs. CFD)

Step 1: Indentify the flow domain & geometry

Step 2: List assumptions, approximations & Boundary Conditions (BCs)

critical ✱✱

Step 3: List all appropriate equations & unknowns

For incompressible flow,

Unknowns	eq's
u, v, w	· cons. of mass (1)
P	· linear momentum (3)
4 unknowns	4 eqs

[compressible flow → add ρ, T as unknowns, add energy eq & an eq. of state as eq's]

· Step 4 Solve eqs (integrate the diff. eq's)

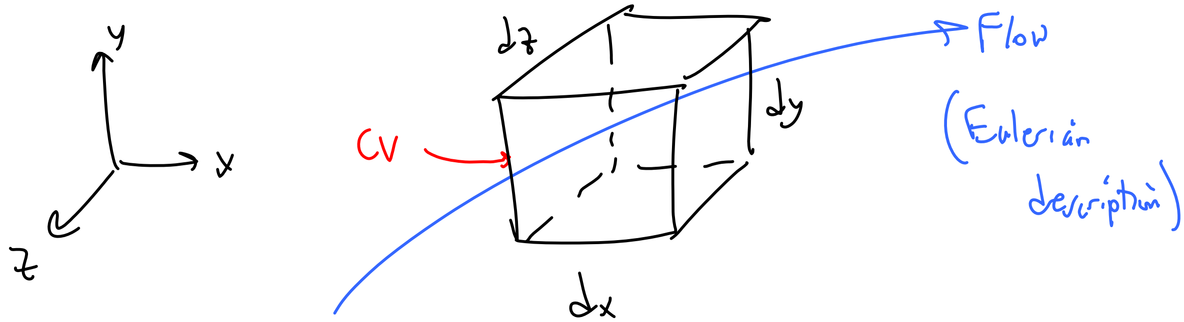
· Step 5 Apply BC's

These 2 steps are sometimes reversed or done simultaneously

· Step 6 Verify results → make sure answers make sense & satisfy eq's &, BC's.

C. Conservation of Mass → The continuity eq

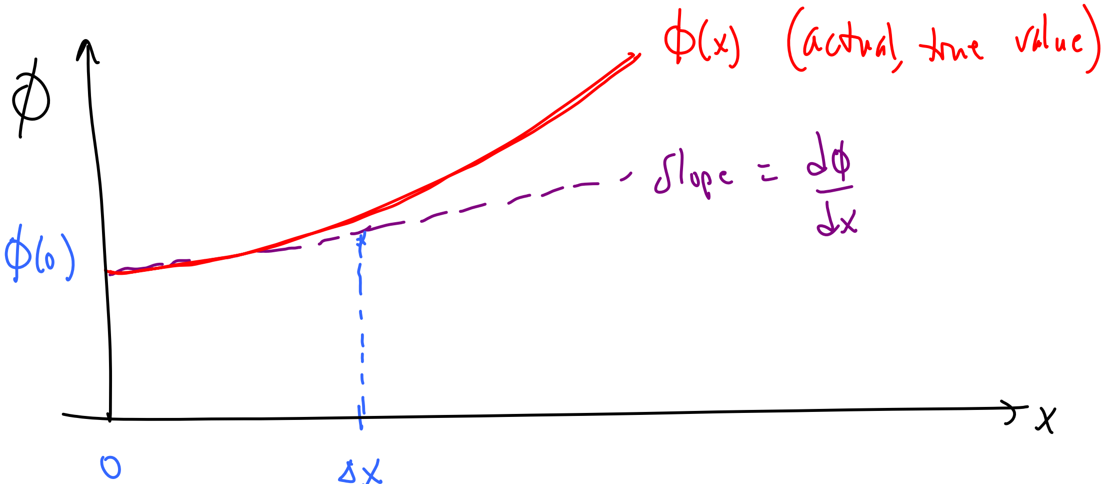
1. Derivation → look at an infinitesimal control volume in the fluid



Imagine that this CV shrinks to a point

$$dx,\ dy,\ dz \to 0$$

· Use Taylor series expansions to study mass flow rate through each face of our CV (6 faces)



$$\phi(\Delta x) = \phi(0) + \frac{d\phi}{dx}\Delta x + \cancel{\frac{1}{2!}\frac{d^2\phi}{dx^2}(\Delta x)^2 + \ldots}$$

As $\Delta x \to 0$ we can neglect the higher order terms

Derivation of the Continuity Equation
(Section 9-2, Çengel and Cimbala)

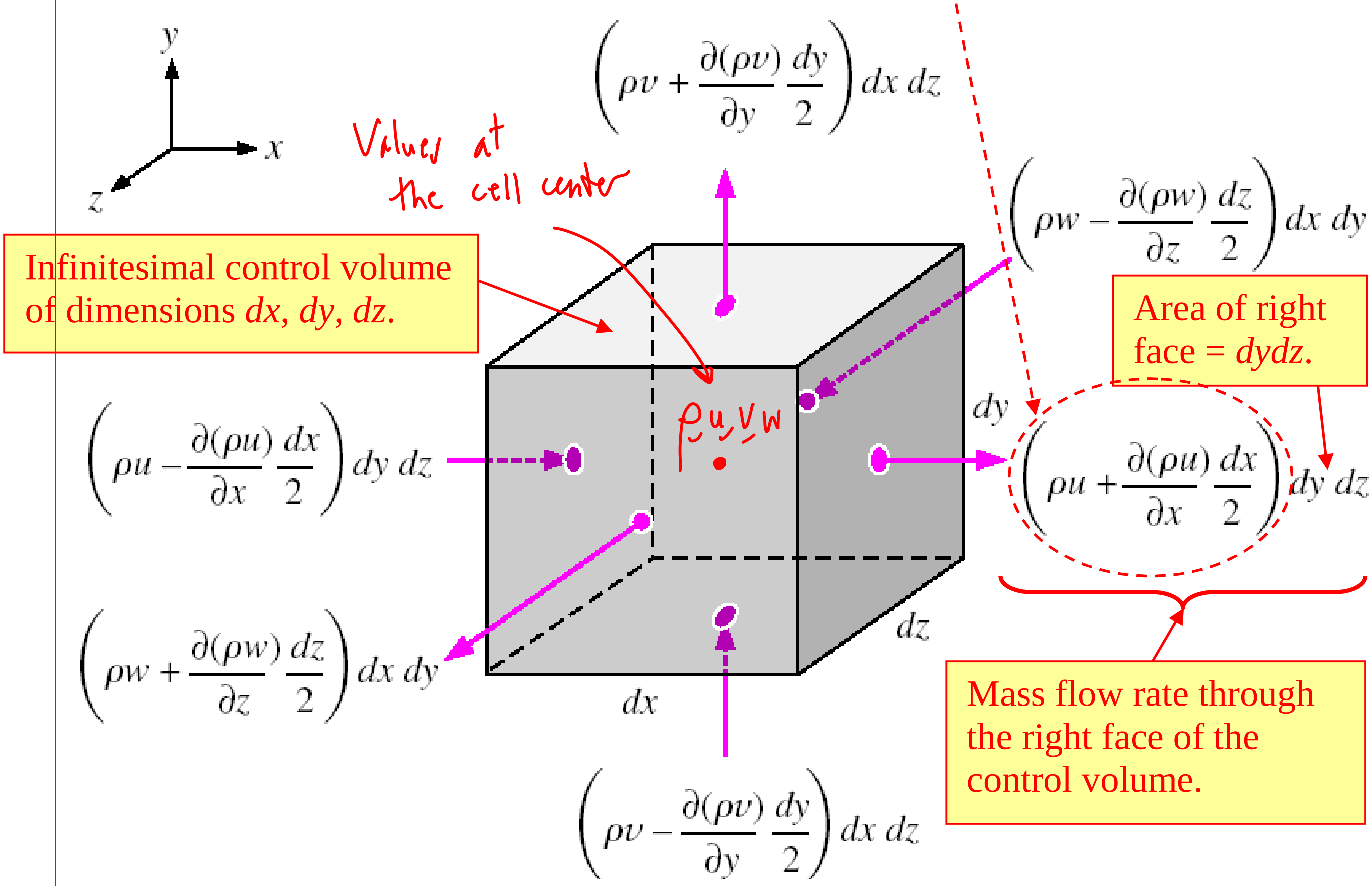
We summarize the second derivation in the text – the one that uses a ***differential control volume***. First, we approximate the mass flow rate into or out of each of the six surfaces of the control volume, using ***Taylor series expansions*** around the center point, where the velocity components and density are u, v, w, and ρ. For example, at the right face,

This is our ϕ in Taylor series

Ignore terms higher than order dx.

$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \cdots \qquad \textbf{(9–6)}$$

The mass flow rate through each face is equal to ρ times the normal component of velocity through the face times the area of the face. We show the mass flow rate through all six faces in the diagram below (Figure 9-5 in the text):



Next, we add up all the mass flow rates through all six faces of the control volume in order to generate the general (unsteady, incompressible) ***continuity equation***:

Net mass flow rate into CV: all the *positive* mass flow rates (into CV)

$$\sum_{\text{in}} \dot{m} \cong \underbrace{\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{left face}} + \underbrace{\left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{bottom face}} + \underbrace{\left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{rear face}}$$

Net mass flow rate out of CV: all the *negative* mass flow rates (out of CV)

$$\sum_{\text{out}} \dot{m} \cong \underbrace{\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{right face}} + \underbrace{\left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{top face}} + \underbrace{\left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{front face}}$$

We plug these into the integral conservation of mass equation for our control volume:

$$\boxed{\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, dV} = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m} \tag{9–2}$$

This term is approximated at the center of the tiny control volume, i.e.,

$$\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, dV \cong \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

The conservation of mass equation (Eq. 9-2) thus becomes

$$\frac{\partial \rho}{\partial t} dx\, dy\, dz = -\frac{\partial(\rho u)}{\partial x} dx\, dy\, dz - \frac{\partial(\rho v)}{\partial y} dx\, dy\, dz - \frac{\partial(\rho w)}{\partial z} dx\, dy\, dz$$

Dividing through by the volume of the control volume, *dxdydz*, yields

Continuity equation in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{9–8}$$

Valid for unsteady, compressible flow

Finally, we apply the definition of the ***divergence*** of a vector, i.e.,

$$\vec{\nabla} \cdot \vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} \quad \text{where} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \quad \text{and} \quad \vec{G} = \left(G_x, G_y, G_z\right)$$

Letting $\vec{G} = \rho\vec{V}$ in the above equation, where $\vec{V} = (u, v, w)$, Eq. 9-8 is re-written as

Continuity equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0} \tag{9–5}$$

(1)

General form, valid for unsteady, compressible flow

2. Simplifications

a. Steady but compressible $\rightarrow \frac{\partial}{\partial t}(\ ) = 0$ (anything), $\therefore \frac{\partial \rho}{\partial t} = 0$

$$\therefore (1) \rightarrow \boxed{\vec{\nabla} \cdot (\rho \vec{V}) = 0} \qquad (2)$$

b. Incompressible, but unsteady

$\rho = \text{constant} \longrightarrow \therefore \frac{\partial \rho}{\partial (\ )} = 0$ (anything), $\therefore \frac{\partial \rho}{\partial t} = 0$

(1) becomes $\vec{\nabla} \cdot (\rho \vec{V}) = 0$ but ρ = constant.

$$\therefore \rho(\vec{\nabla} \cdot \vec{V}) = 0 \qquad \div \rho \rightarrow \boxed{\vec{\nabla} \cdot \vec{V} = 0} \ (3)$$

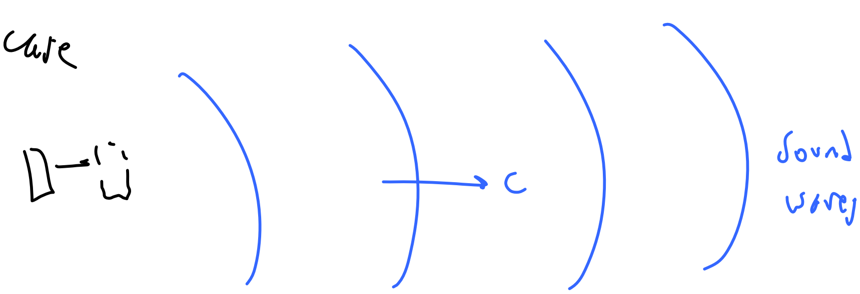
Continuity eq. has no time term even for unsteady flow

Implies speed of sound = infinite

disturbance

disturbance is felt instantaneously everywhere in the flow

real case



In this course, most of our problems are incompressible

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$ ← Incompressible continuity eq. ☆

· Cartesian coord. $\vec{x} = (x, y, z)$ $\vec{V} = (u, v, w)$ $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

☆ Most useful form for us

· Cylindrical coord: $\vec{x} = (r, \theta, z)$ $\vec{V} = (u_r, u_\theta, u_z)$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}(r\, u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0}$$

☆ Second most useful form for us

Valid for steady or unsteady incompressible flow

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3. Examples

e.g. Given:

$$\boxed{\begin{aligned} u &= a(x^2 y + y^2) \\ v &= b\, y^2 x \\ w &= c \end{aligned}}$$

→ $\frac{\partial u}{\partial x} = 2axy$

→ $\frac{\partial v}{\partial y} = 2bxy$

→ $\frac{\partial w}{\partial z} = 0$

To do: Under what conditions is this velocity field a valid steady, incompressible velocity field?

Sol'n: Must satisfy the continuity eq.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$2axy + 2bxy + 0 = 0$$

Solve → $\boxed{a = -b}$ ← ANSWER ☆

Criterion for this velocity field to be a valid steady, incomp. velocity field

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Example Given $u = ax + b$, $w = 0$ (2-D)

To do: Find v such that the flow is steady & incompressible in the x-y plane (2-D)

Sol'n: Continuity eq. must be satisfied:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$$

(2-D)

$$a + \frac{\partial v}{\partial y} = 0 \rightarrow \boxed{\frac{\partial v}{\partial y} = -a}$$

Integrate: This is a partial integration w.r.t. y, but $v = \text{fnc}(x, y)$

$$\boxed{☆\ v = -ay + f(x)}$$

← Any $f(x)$ will work since $\frac{\partial}{\partial y}\left(f(x)\right) = 0$

ANSWER ↗ ☆

Check: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = a - a = 0$ ✓