Today, we will:

- Do another example problem – continuity equation
- Discuss the stream function and its physical significance, and do some examples

3. Examples (continued)

Given: • 2-D, steady, incomp flow in r-θ plane

• $u_\theta = C\theta$ ($w=0$)

To do: Find u_r

Soln: Apply continuity eq.

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{C} + \cancelto{2D}{\frac{\partial u_z}{\partial z}} = 0$$

$\times r \Rightarrow$ $$\frac{\partial}{\partial r}(r u_r) = -C$$

int. w.r.t. r $$r u_r = -Cr + f(\theta)$$

$\div r \rightarrow$ $$\boxed{u_r = -C + \frac{f(\theta)}{r}} \quad \star$$

Check or verify → Cont: $$\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_\theta}{\partial \theta} \stackrel{?}{=} 0$$

$$\frac{\partial}{\partial r}\left(-Cr + f(\theta)\right) + \frac{\partial u_\theta}{\partial \theta} = 0$$

$$-C + C = 0 \quad \checkmark$$

D. The Stream Function, ψ (Sec 9-3)

1. Definition → for 2-D incomp flow in the x-y plane

(1)

- Cont. eq. $\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$

Definition of ψ

$$\boxed{u = \frac{\partial \psi}{\partial y}, \quad v = \frac{-\partial \psi}{\partial x}}$$

- Define ψ by its derivatives

$$\psi = \psi(x,y)$$

$$\boxed{\psi = \text{stream function}}$$

- Eq. (1) becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Exactly satisfied if $\psi(x,y)$ is smooth & continuous

∴ For a given flow field, if we can define a smooth, continuous $\psi(x,y)$, then continuity is automatically satisfied.

Also, we can easily calculate u & v for a known $\psi(x,y)$

<u>Example</u>: Suppose $\psi(x,y) = ax^3 + byx$

$$u = \frac{\partial \psi}{\partial y} = bx \qquad v = \frac{-\partial \psi}{\partial x} = \underbrace{-3ax^2 - by}$$

Check continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

$$b - b = 0$$

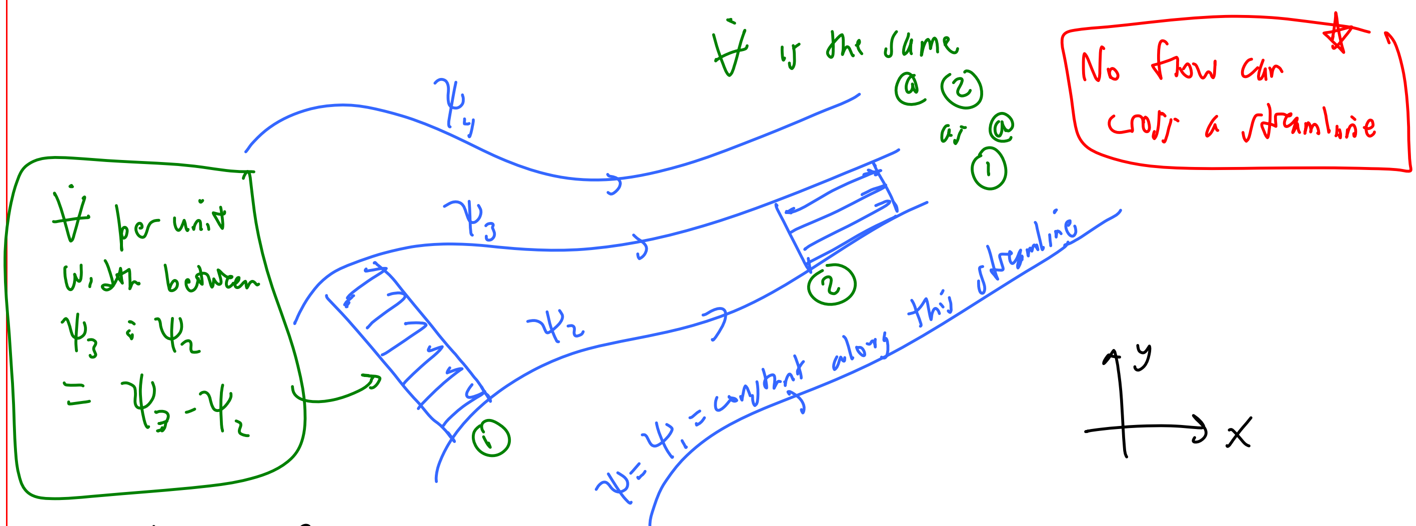
✓ Continuity is <u>satisfied</u>

2. Physical Significance of Ψ:

a. Streamlines $\rightarrow$ $\boxed{\text{Lines of constant } \Psi \text{ are streamlines of the flow!}}$

(see text for proof)

or $\boxed{\Psi = \text{constant along a streamline}}$



b. Volume flow rate

$\boxed{\text{The difference in } \Psi \text{ from one streamline to another is } \underline{\text{equal}} \text{ to the volume flow rate per unit width (into the page - z direction) between those two streamlines}}$

As streamlines converge, the flow velocity must increase to keep the volume flow rate constant between the streamlines

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3. Examples $\rightarrow$ see Examples 9-8 through 9-11

Given: $\boxed{\begin{aligned} u &= x^2 \\ v &= -2xy - 1 \end{aligned}}$ · Steady, 2-D, incomp. flow in x-y plane

To do: (a) Calculate $\psi(x,y)$

(b) Plot streamlines – what kind of flow is this?

Soln: (a) First check continuity. [If continuity is not satisfied then we stop! ψ is not defined]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2x - 2x = 0 \checkmark$$

· Pick one of the definitions of ψ: $\frac{\partial \psi}{\partial y} = u = x^2$ ← Given

· Integrate: $\boxed{\psi = x^2 y + f(x)}$

· Now use the other eq. for ψ: $\frac{\partial \psi}{\partial x} = -v = 2xy + 1$ } equate

· Take x-derivative of ψ: $\frac{\partial \psi}{\partial x} = 2xy + f'(x)$

· Equate the 2 eqs for $\frac{\partial \psi}{\partial x}$: $2\cancel{x}y + 1 = 2\cancel{x}y + f'(x)$

$\therefore f'(x) = 1$

· Integrate to get $f(x)$ → total integration, not a partial

$$\underline{f(x) = x + C}$$

· Finally $\boxed{\psi(x,y) = x^2 y + x + C}$ ☆ C = arbitrary constant

(b) Plot streamlines ⟶ means plot curves of constant ψ

(I did it in EES) → see next page

I used EES to plot 3 streamlines, ψ = 3, 2, and 1:

```
psi = x^2*y+x+C
C=0
psi=1
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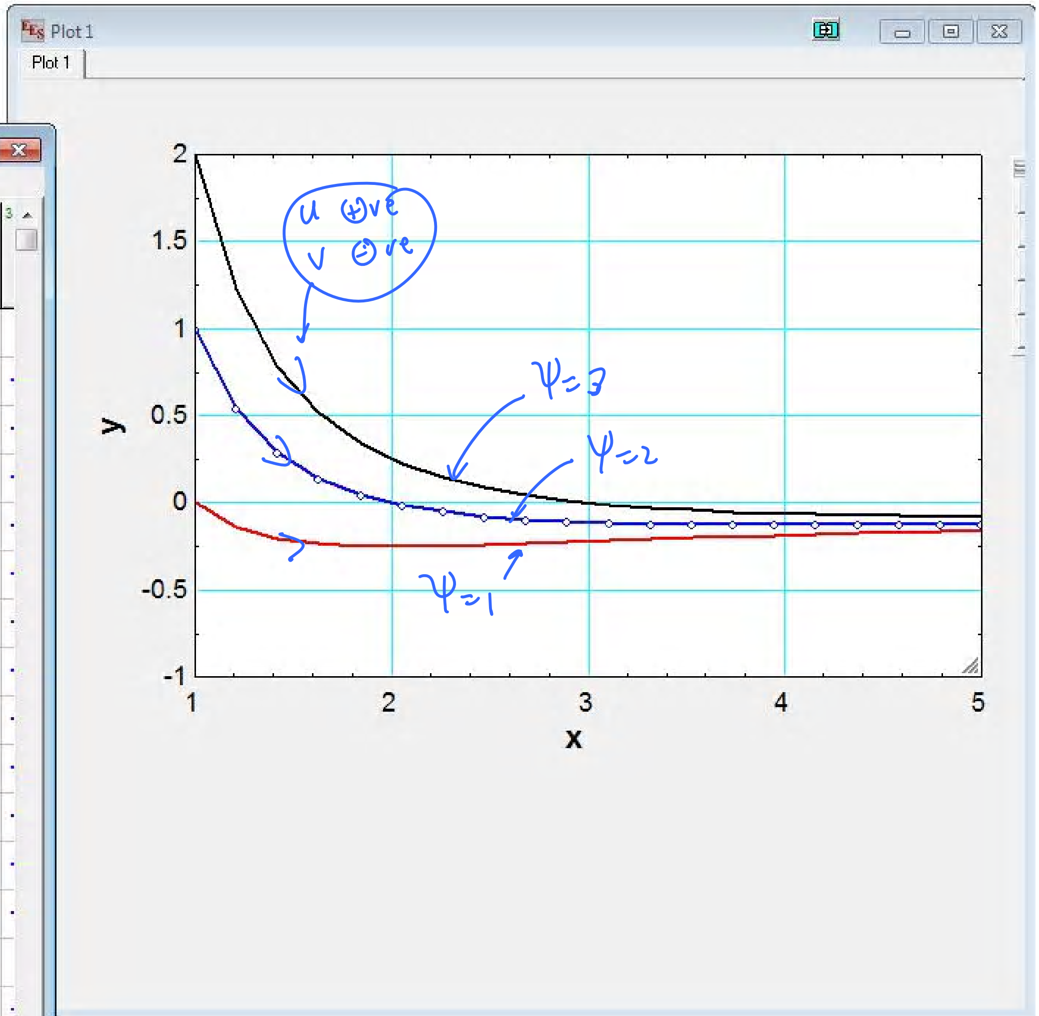
Parametric Table — Table 1

1..20	ψ	x
Run 1	1	1
Run 2	1	1.211
Run 3	1	1.421
Run 4	1	1.632
Run 5	1	1.842
Run 6	1	2.053
Run 7	1	2.263
Run 8	1	2.474
Run 9	1	2.684
Run 10	1	2.895
Run 11	1	3.105
Run 12	1	3.316
Run 13	1	3.526
Run 14	1	3.737
Run 15	1	3.947
Run 16	1	4.158

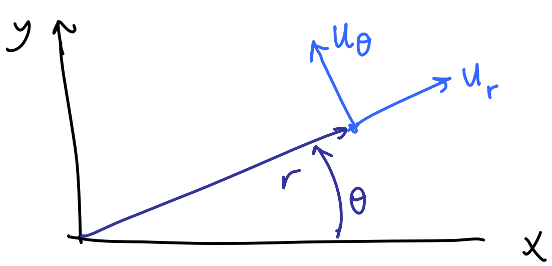


4. Stream Function in Cylindrical Coordinates (r, θ, z)

2 possibilities → • planar flow (in r-θ plane) (same as x-y plane)

• axisymmetric flow (in r-z plane)

a. Planar flow   no z dependence



Continuity eq. $\quad \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) + \cancel{\frac{\partial u_z}{\partial z}} = 0$ 2-D planar

$\times r$: $\quad \boxed{\frac{\partial}{\partial r}(r u_r) + \frac{\partial}{\partial \theta}(u_\theta) = 0}$

☆ Defn of ψ in cyl. coord.

Define $\psi(r,\theta)$ as $\quad \boxed{u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad u_\theta = -\frac{\partial \psi}{\partial r}}$

check continuity:

$$\frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial \theta}\right) - \frac{\partial}{\partial \theta}\left(\frac{\partial \psi}{\partial r}\right)$$

$$= \frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{\partial^2 \psi}{\partial \theta \partial r} = 0$$

For a smooth continuous function $\psi(r,\theta)$, this eq. is exactly satisfied

☆ **Continuity is exactly satisfied by our definition of ψ** ☆

b. Axisymmetric flow → flow in r-z plane, no θ-dependence

See text for equations for $\psi(r,z)$

Example: Given: · 2-D steady, incomp flow in r-θ plane

· $\boxed{\psi = V_\infty r \sin\theta}$

To do: (a) Calc. u_r & u_θ &, convert to u & v

(b) Sketch streamlines

Sol^n: (a) Defn of Ψ:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r} V_\infty r \cos\theta = V_\infty \cos\theta$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta$$

$$\boxed{\begin{aligned} u_r &= V_\infty \cos\theta \\ u_\theta &= -V_\infty \sin\theta \end{aligned}}$$

• To convert to u & v, see Eq.s 9-11

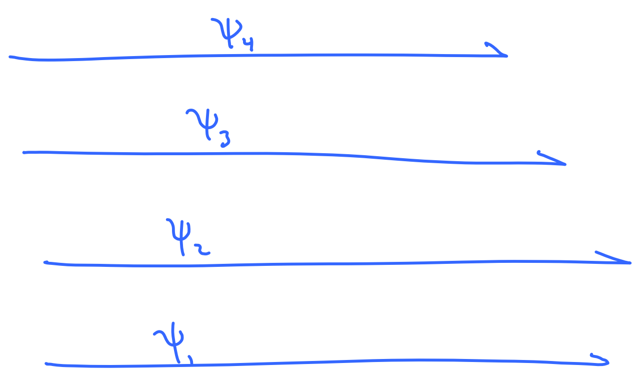
i.e. $x = r\cos\theta$ $\quad y = r\sin\theta$

$$\therefore \psi = V_\infty r \sin\theta = V_\infty y \rightarrow \boxed{\begin{aligned} u &= \frac{\partial \psi}{\partial y} = V_\infty \\ v &= -\frac{\partial \psi}{\partial x} = 0 \end{aligned}}$$

$$\boxed{\psi = V_\infty y}$$

(b) Draw streamlines:

Sol^n: lines of constant ψ = streamlines = lines of constant y



$$\left(\begin{aligned} u &= V_\infty = \text{constant} \\ v &= 0 \end{aligned} \right)$$

So, this is just a uniform flow in the x-direction!