Today, we will:

- Do another example – the stream function.
- Discuss the differential equation for momentum in fluid flow: **The Navier-Stokes eq**.
- Do some example problems – Navier-Stokes equation

EXAMPLE:

Given: • 2-D, steady, incomp. flow in r-θ plane (x-y plane)

• $\boxed{u_r = \frac{c}{r}}$ $u_\theta = 0$

To do: Calc. $\psi(r,\theta)$ & sketch streamlines

Soln: Pick one of the eq's for ψ:

• $\boxed{u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}} = \frac{c}{r}$

(x r) → $\frac{\partial \psi}{\partial \theta} = c$ → integrate: $\psi = c\theta + f(r)$

• Use the other eq for ψ:

$\boxed{u_\theta = -\frac{\partial \psi}{\partial r}} = -f'(r) = 0$ → $f'(r) = 0$

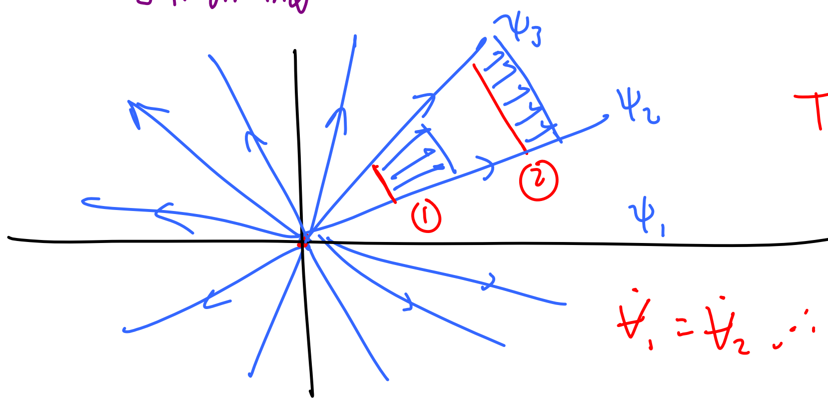
Integrate: $f(r) = \text{const}$

∴ $\boxed{\psi = c\theta + \text{const}}$

Streamlines: → streamlines = lines of constant ψ = lines of const θ

Singularity @ origin

$(V = \infty)$



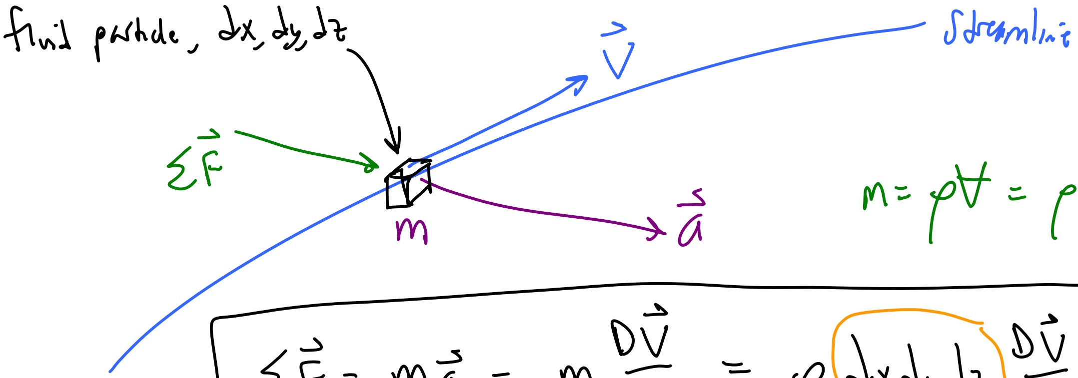
THIS IS A LINE SOURCE

$\dot{\forall}_1 = \dot{\forall}_2 \quad \therefore V_1 > V_2$

E. Cons. of Linear Momentum

1. Derivation → See Text (3 ways in text)

Easiest derivation = from Newton's 2nd Law

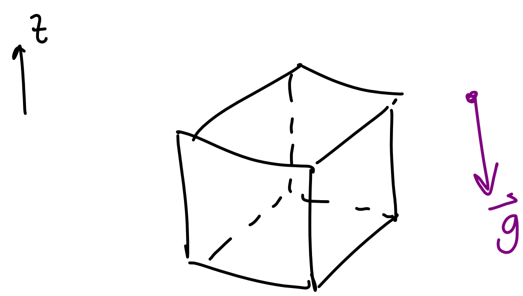


$m = \rho \forall = \rho\, dx\, dy\, dz$

$$\boxed{\sum \vec{F} = m\vec{a} = m\frac{D\vec{V}}{Dt} = \rho\, dx\, dy\, dz\, \frac{D\vec{V}}{Dt}} \quad (1)$$

(material accel from Ch. 4)

following a fluid particle



$\sum \vec{F} = \sum F_{body} + \sum \vec{F}_{surface}$

gravity (weight)

pressure & viscous forces

See text for derivation

$$\sum \vec{F} = \rho \vec{g}\, dx\, dy\, dz + \underbrace{\vec{\nabla} \cdot \sigma_{ij}}_{\text{vector}}\, dx\, dy\, dz \quad (2)$$

σ_{ij} = the stress tensor (pressure stresses + viscous stresses)

Combine (1) & (2) → $\boxed{\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} + \vec{\nabla} \cdot \sigma_{ij}}$ CAUCHY'S EQ.

Cauchy's eq. is not useful unless we have an eq. for σ_{ij}

↓ Very general – valid for any fluid.

☆ Lets consider ONLY Newtonian fluids in this course

↓ we get Navier-Stoke eq.

(a special case of the Cauchy eq)

☆ Newtonian fluid → shear stress is linearly proportional to shear strain rate ☆

See derivation in text, next page →

2. Applications of the N-S eq.

Two main applications:

- Determine pressure field for a known velocity field
- Solve fluid flow problems from scratch

EXAMPLES → See text.

Derivation of the Navier-Stokes Equation
(Section 9-5, Çengel and Cimbala)

We begin with the general differential equation for conservation of linear momentum, i.e., ***Cauchy's equation***, which is valid for any kind of fluid,

Cauchy's equation:

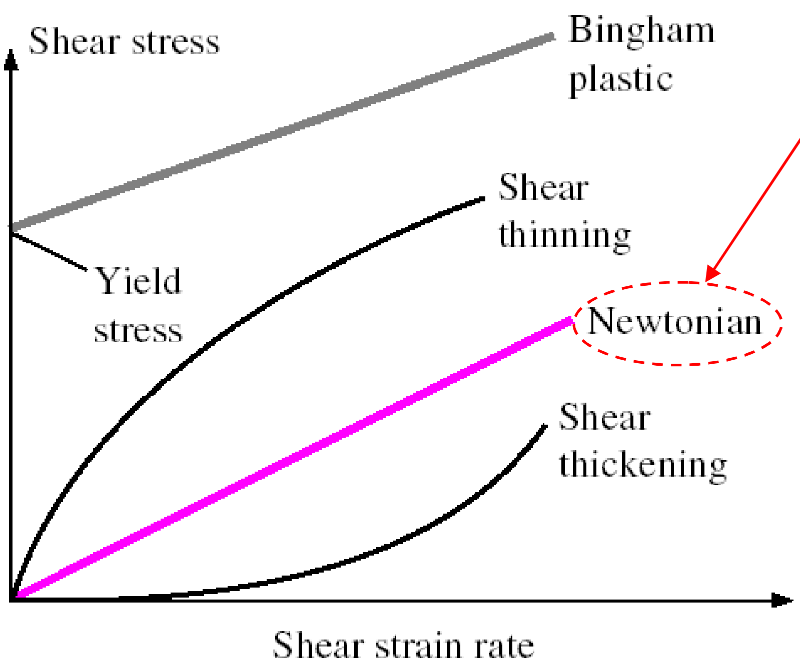
$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij} \qquad \textbf{(9–50)}$$

Stress tensor

The problem is that the stress tensor σ_{ij} needs to be written in terms of the primary unknowns in the problem in order for Cauchy's equation to be useful to us. The equations that relate σ_{ij} to other variables in the problem – velocity, pressure, and fluid properties – are called ***constitutive equations***. There are different constitutive equations for different kinds of fluids.

↳ unknowns in the problem: u, v, w, p (incomp. flow)

Types of fluids:



For ***Newtonian fluids****, the shear stress is linearly proportional to the shear strain rate.*

Examples of Newtonian fluids: water, air, oil, gasoline, most other common fluids.

FIGURE 9–~~37~~ 38

Rheological behavior of fluids—shear stress as a function of shear strain rate.

Some examples of non-Newtonian fluids:

- Paint (***shear thinning*** or ***pseudo-plastic***)
- Toothpaste (***Bingham plastic***)
- Quicksand (***shear thickening*** or ***dilatant***).

We consider only Newtonian fluids in this course.

For ***Newtonian fluids*** (see text for derivation), it turns out that

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & 2\mu\dfrac{\partial v}{\partial y} & \mu\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & 2\mu\dfrac{\partial w}{\partial z} \end{pmatrix} \qquad \text{(9–57)}$$

We have achieved our goal of writing σ_{ij} in terms of pressure P, velocity components u, v , and w, and fluid viscosity μ.

Now we plug this expression for the stress tensor σ_{ij} into Cauchy's equation. The result is the famous ***Navier-Stokes equation***, shown here for incompressible flow.

Incompressible Navier–Stokes equation:

Navier-Stokes equation:

$$\boxed{\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}} \qquad \text{(9–60)}$$

Foundation of fluid mechanics

To solve fluid flow problems, we need both the continuity equation and the Navier-Stokes equation. Since it is a vector equation, the Navier-Stokes equation is usually split into three components in order to solve fluid flow problems. In Cartesian coordinates,

Incompressible continuity equation:

4 eqs, 4 unknowns (u, v, w, P)

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \qquad \text{(9–61a)}$$

Cont.

x-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \text{(9–61b)}$$

y-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(9–61c)}$$

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z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad \text{(9–61d)}$$

Example: Given: • 2-D ~~steady~~, incomp flow

• $\boxed{u = x^2 - y^2}$ • $w = 0$ (2-D)

• $\boxed{v = -2xy}$ • ignore gravity (no gravity in x & y directions)

To do: Calculate $p(x,y)$

Sol'n: ☆☆☆ First check if continuity is satisfied

↓ if not, stop

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0 \quad \text{2D}$$

$$2x - 2x = 0$$

✓ Yes

• N-S eq.

x-comp:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \cancel{\rho g_x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

steady 2-D (g = in z-dir) $\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = -2y$ 2-D

$$\rho\left((x^2-y^2)(2x) + (-2xy)(-2y)\right) = -\frac{\partial p}{\partial x} + \mu\underbrace{(2-2)}_{0}$$

$$\boxed{\frac{\partial p}{\partial x} = -2\rho\left(x^3 + xy^2\right)} \quad (1)$$

y-comp: Do on your own → get $\boxed{\frac{\partial p}{\partial y} = -2\rho\left(y^3 + x^2 y\right)}$ (2)

recall for a smooth continuous fnc.,

$$\frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x}$$

Check this by plugging into (1) & (2) → verify this on your own

- Integrate to get $P(x,y)$
- Pick x-comp. → Eq (1) → int. (1) w.r.t. x (partial int)

$$P = -2\rho\left(\frac{x^4}{4} + \frac{x^2y^2}{2}\right) + f(y) \qquad (3)$$

- Take y-deriv: $$\frac{\partial P}{\partial y} = -2\rho\left(x^2y\right) + f'(y)$$

Compare this to Eq. (2) above & equate:

$$f'(y) = -2\rho y^3$$

Integrate: $$f(y) = -\frac{1}{2}\rho y^4 + C$$

$$\therefore \quad \boxed{P(x,y) = -\frac{1}{2}\rho\left(x^4 + 2x^2y^2 + y^4\right) + C}$$

ANSWER

NOTICE – THERE IS ALWAYS AN ARBITRARY CONSTANT

WHY? Fluid flow is controlled by changes in pressure. The absolute value of pressure is not important, only pressure gradients are important