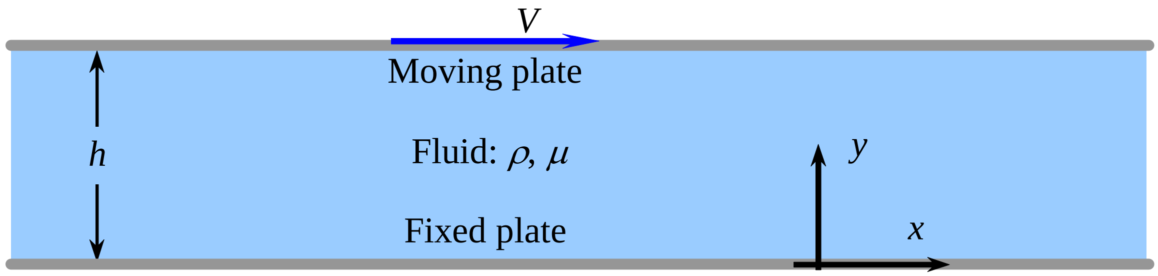
Today, we will:

- Do some example problems – Exact solutions ~~to~~ of the Navier-Stokes equation

Example Problem – Exact Solution for Couette Flow

Given: Steady, incompressible, fully developed laminar flow in the *x*-*y* plane between two infinite parallel plates.



Assumptions and approximations:

1. The flow is steady [$\partial/\partial t$ of anything = 0].
2. The flow is two-dimensional in the *x*-*y* plane [$\partial/\partial z$ of anything = 0, w = 0].
3. Gravity effects are negligible or ignored.
4. The flow is fully developed [$\partial/\partial x$ of any velocity = 0; velocity does not change with x].
5. Pressure is constant everywhere. $\rightarrow \frac{\partial P}{\partial(\text{anything})} = 0$

To do: Calculate the velocity field.

Solution: [to be done in class]

- Step 1 Set up problem w/ sketch ✓
- Step 2 List assumptions & approximations — I like to number them. ✓

List BCs

$$\boxed{\begin{array}{l}(1)\ @\ y=0,\ u=v=0\\ (2)\ @\ y=h,\ u=V,\ v=0\end{array}}$$

- Step 3 Simplify diff. eqs as much as possible

Continuity:

$$\underset{(4)}{\cancel{\frac{\partial u}{\partial x}}} + \frac{\partial v}{\partial y} + \underset{(2)}{\cancel{\frac{\partial w}{\partial z}}} = 0 \rightarrow \boxed{\frac{\partial v}{\partial y} = 0} \quad (1)$$

- x-mom

$$\rho\left(\underset{(1)}{\cancel{\frac{\partial u}{\partial t}}} + u\underset{(4)}{\cancel{\frac{\partial u}{\partial x}}} + v\frac{\partial u}{\partial y} + w\underset{(2)}{\cancel{\frac{\partial u}{\partial z}}}\right) = -\underset{(5)}{\cancel{\frac{\partial P}{\partial x}}} + \rho\underset{(3)}{\cancel{g_x}} + \mu\left(\underset{\text{(See below)}}{\cancel{\frac{\partial^2 u}{\partial x^2}}} + \frac{\partial^2 u}{\partial y^2} + \underset{(2)}{\cancel{\frac{\partial^2 u}{\partial z^2}}}\right)$$

[Since $\frac{\partial u}{\partial x} = 0$ everywhere, then $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(0) = 0$]

$$\boxed{\rho v \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2}} \qquad (2)$$

y-mom: $$\rho\left(\cancel{\frac{\partial v}{\partial t}} + u\cancel{\frac{\partial v}{\partial x}} + v\cancel{\frac{\partial v}{\partial y}} + w\cancel{\frac{\partial v}{\partial z}}\right) = -\cancel{\frac{\partial P}{\partial y}} + \rho \cancel{g_y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

(1) (4) Eq. (1) (2) (5) (3) (4) Eq. (1) (2)

Since $\frac{\partial v}{\partial y} = 0$, then $\frac{\partial^2 v}{\partial y^2} = 0$

$$\boxed{0 = 0}$$

y-mom eq. is identically satisfied for our problem

z-mom $\rightarrow$ $\boxed{0 = 0}$

- Step 4 Solve (integrate) the eq's
- Step 5 Apply BC's

usually do these simultaneously

Eq. (1) $\frac{\partial v}{\partial y} = 0 \rightarrow \underline{v = f(x)}$

But, we know that $\frac{\partial v}{\partial x} = 0$ by assump (4)

$f'(x) = 0$

$f(x) = $ const

$\therefore$ $\boxed{v = \text{const}}$ everywhere Apply BC $\rightarrow$ @ $y = 0$, $v = 0$ $\Rightarrow$ $\therefore$ $\boxed{v = 0 \text{ everywhere}}$

- Eq. (2) $\rightarrow$ $\rho v \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} \rightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = 0}$ (3)

0

But:
- $u \neq$ fnc. (x) by Approx. (4)
- $u \neq$ fnc (t) by Approx (1)
- $u \neq$ fnc (z) by " (2)

$\therefore$ $u = u(x, y, z, t)$ in general

Here, $\underline{u = u(y) \text{ only}}$

Eq (3) can be simplified as $\boxed{\dfrac{d^2u}{dy^2}=0}$ (4)

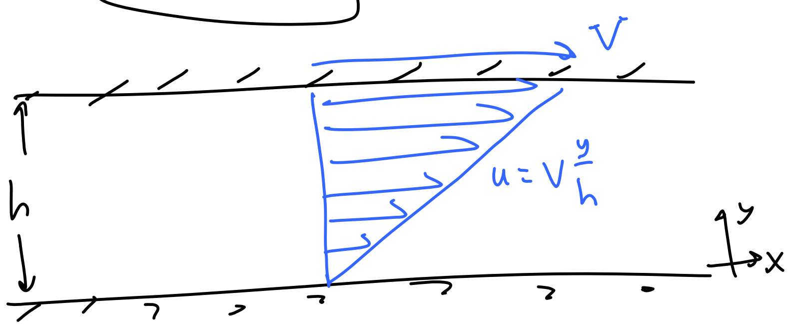
· Integrate (twice) $\dfrac{du}{dy}=C_1 \rightarrow u = C_1 y + C_2$

· Apply BC's to solve for C_1 & C_2

@ $y=0$, $u=0 \longrightarrow 0 = C_1(0) + C_2 \rightarrow \boxed{C_2=0}$

@ $y=h$, $u=V \longrightarrow V = C_1(h) + \cancelto{0}{C_2} \rightarrow \boxed{C_1=\dfrac{V}{h}}$

$\therefore \boxed{u = V\dfrac{y}{h}}$ ANSWER !



Couette Flow

See text for other example problems.

Example – Laminar Pipe Flow; an Exact Solution of the Navier-Stokes Equation (Example 9-18, Çengel and Cimbala)

Note: This is a *classic* problem in fluid mechanics.

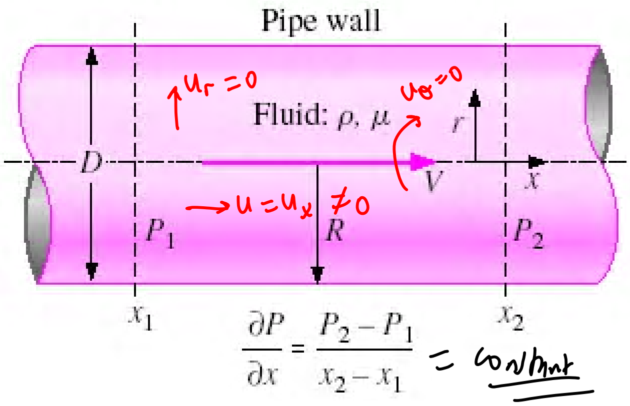


FIGURE 9-71
Geometry of Example 9–18: steady laminar flow in a long round pipe with an applied pressure gradient $\partial P/\partial x$ pushing fluid through the pipe. The pressure gradient is usually caused by a pump and/or gravity.

EXAMPLE 9–18 Fully Developed Flow in a Round Pipe—Poiseuille Flow

Consider steady, incompressible, laminar flow of a Newtonian fluid in an infinitely long round pipe of diameter D or radius $R = D/2$ (Fig. 9–69). We ignore the effects of gravity. A constant pressure gradient $\partial P/\partial x$ is applied in the x-direction,

Fully developed flow

Applied pressure gradient:
$$\frac{\partial P}{\partial x} = \frac{P_2 - P_1}{x_2 - x_1} = \text{constant} \quad \textbf{(1)}$$

where x_1 and x_2 are two arbitrary locations along the x-axis, and P_1 and P_2 are the pressures at those two locations. Note that we adopt a modified cylindrical coordinate system here with x instead of z for the axial component, namely, (r, θ, x) and (u_r, u_θ, u). Derive an expression for the velocity field inside the pipe and estimate the viscous shear force per unit surface area acting on the pipe wall.

It is good practice to *number* the assumptions.

SOLUTION For flow inside a round pipe we are to calculate the velocity field, and then estimate the viscous shear stress acting on the pipe wall.

Assumptions **1** The pipe is infinitely long in the x-direction. **2** The flow is steady (all partial time derivatives are zero). **3** This is a parallel flow (the r-component of velocity, u_r, is zero). **4** The fluid is incompressible and Newtonian with constant properties, and the flow is laminar. **5** A constant-pressure gradient is applied in the x-direction such that pressure changes linearly with respect to x according to Eq. 1. **6** The velocity field is axisymmetric with no swirl, implying that $u_\theta = 0$ and all partial derivatives with respect to θ are zero. **7** We ignore the effects of gravity.

Analysis To obtain the velocity field, we follow the step-by-step procedure outlined in Fig. 9–50.

Step 1 *Lay out the problem and the geometry.* See Fig. 9–69.

Step 2 *List assumptions and boundary conditions.* We have listed seven assumptions. The first boundary condition comes from imposing the no-slip condition at the pipe wall: (1) at $r = R$, $\vec{V} = 0$. The second boundary condition comes from the fact that the centerline of the pipe is an axis of symmetry: (2) at $r = 0$, $du/dr = 0$.

Step 3 *Write out and simplify the differential equations.* We start with the incompressible continuity equation in cylindrical coordinates, a modified version of Eq. 9–62a,

$$\underbrace{\cancel{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}}_{\text{assumption 6}} + \frac{\partial u}{\partial x} = 0 \qquad \rightarrow \qquad \frac{\partial u}{\partial x} = 0 \tag{2}$$

Equation 2 tells us that u is not a function of x. In other words, it doesn't matter where we place our origin—the flow is the same at any x-location. This can also be inferred directly from assumption 1, which tells us that there is nothing special about any x-location since the pipe is infinite in length—the flow is fully developed. Furthermore, since u is not a function of time (assumption 2) or θ (assumption 6), we conclude that u is at most a function of r,

Result of continuity: $$\boxed{u = u(r) \text{ only}} \tag{3}$$

This is a tremendous simplification, and allows us to solve the problem analytically!

We now simplify the axial momentum equation (a modified version of Eq. 9–62d) as far as possible:

When terms drop out, I like to show why, as I do here (for clarity), using the numbered assumptions for brevity.

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{assumption 2}} + \underbrace{\cancel{u_r\frac{\partial u}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u}{\partial \theta}}}_{\text{assumption 6}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\text{continuity}}\right)$$

$$= -\frac{\partial P}{\partial x} + \underbrace{\cancel{\rho g_x}}_{\text{assumption 7}} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}}_{\text{assumption 6}} + \underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{continuity}}\right)$$

or

$$\boxed{\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x}} \tag{4}$$

As in Examples 9–15 through 9–17, the material acceleration (entire left side of the *x*-momentum equation) is zero, implying that fluid particles are not accelerating at all in this flow field, and linearizing the Navier–Stokes equation (Fig. 9–70). We have replaced the partial derivative operators for the *u*-derivatives with total derivative operators because of Eq. 3.

In similar fashion, every term in the *r*-momentum equation (Eq. 9–62b) except the pressure gradient term is zero, forcing that lone term to also be zero,

r-momentum:
$$\frac{\partial P}{\partial r} = 0 \tag{5}$$

In other words, *P* is not a function of *r*. Since *P* is also not a function of time (assumption 2) or θ (assumption 6), *P* can be at most a function of *x*,

Result of r-momentum:
$$P = P(x) \text{ only} \tag{6}$$

Therefore, we can replace the partial derivative operator for the pressure gradient in Eq. 4 by the total derivative operator since *P* varies only with *x*. Finally, all terms of the θ-component of the Navier–Stokes equation (Eq. 9–62c) go to zero.

Step 4 *Solve the differential equations.* Continuity and *r*-momentum have already been "solved," resulting in Eqs. 3 and 6, respectively. The θ-momentum equation has vanished, and thus we are left with Eq. 4 (*x*-momentum). After multiplying both sides by *r*, we integrate once to obtain

use $\frac{du}{dr}$ instead of $\frac{\partial u}{\partial r}$ $\rightarrow$
$$r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1 \tag{7}$$

where C_1 is a constant of integration. Note that the pressure gradient *dP*/*dx* is a constant here. Dividing both sides of Eq. 7 by *r*, we integrate a second time to get

$$\boxed{u = \frac{r^2}{4\mu}\frac{dP}{dx} + C_1 \ln r + C_2} \tag{8}$$

where C_2 is a second constant of integration.

Notice that there are *two* constants of integration, since we had to integrate *twice*. Equation 8 is the solution we are looking for, except we need to determine the two constants of integration C_1 and C_2.

BC's come in here to calc. C_1 & C_2

Step 5 *Apply boundary conditions.* First, we apply boundary condition (2) to Eq. 7, NOTE: We apply BC(2) to Eq 7, not Eq. 8 since $\ln(0) = \infty$

Boundary condition (2): $0 = 0 + C_1 \quad \rightarrow \quad \boxed{C_1 = 0}$

An alternative way to interpret this boundary condition is that u must remain finite at the centerline of the pipe. This is possible only if constant C_1 is equal to 0, since ln(0) is undefined in Eq. 8. Now we apply boundary condition (1),

@ r=R u=0

Boundary condition (1): $u = \frac{R^2}{4\mu}\frac{dP}{dx} + 0 + C_2 = 0 \quad \rightarrow \quad C_2 = -\frac{R^2}{4\mu}\frac{dP}{dx}$

Finally, Eq. 7 becomes

Axial velocity: $$\boxed{u = \frac{1}{4\mu}\frac{dP}{dx}(r^2 - R^2)} \qquad (9)$$

Answer

The axial velocity profile is thus in the shape of a paraboloid, as sketched in Fig. 9–71.

Step 6 *Verify the results.* You can verify that all the differential equations and boundary conditions are satisfied.

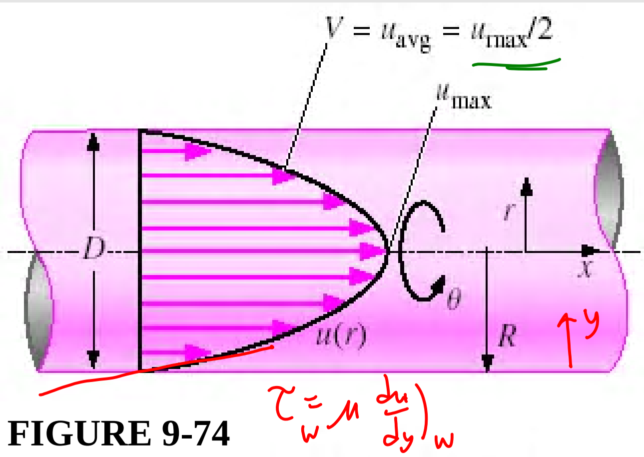


FIGURE 9-74

Axial velocity profile of Example 9–18: steady laminar flow in a long round pipe with an applied constant-pressure gradient dP/dx pushing fluid through the pipe.

For turbulent pipe flow we cannot solve it!

- unsteady
- chaotic
- eddies in flow
- all terms remain in the N-S eq.

[Leads us into CFD Computational Fluid Dynamics — Ch. 15]