Today, we will:

- Begin discussion about Chapter 10 – Approximate solutions of the N-S equation
- Show how to nondimensionalize the equations of motion
- Discuss creeping flow (flow at very low Reynolds number)

VIII. APPROXIMATE SOLUTIONS OF THE NAVIER-STOKES EQUATION

A. Introduction

We have three ways to solve the differential equations of fluid flow:

1. Analytically (Chapter 9) [solve exactly, but only for very simple problems]
2. Numerically (Chapter 15) [use CFD on a computer to solve for thousands of cells]
3. Approximately (Chapter 10) [ignore some terms in the N-S equation, then solve]
   ↳ Simplify the eqs FIRST, before attempting to solve them

B. Nondimensionalization of the Equations of Motion

Goal – To re-write the eqs in a form such that we can compare the order of magnitude of the various terms

Consider incompressible flow only

1. Continuity Eq. $\boxed{\vec{\nabla}\cdot\vec{V}=0}$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\vec{x} = x\vec{i} + y\vec{j} + z\vec{k}$$

• Introduce scaling parameters

• Let L = some characteristic length in the flow

let, define $\boxed{\vec{x}^* = \frac{\vec{x}}{L}} \rightarrow \boxed{x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad z^* = \frac{z}{L}}$ nondimensional

• Let V = some characteristic velocity scale in the flow

Define $\boxed{\vec{V}^* = \frac{\vec{V}}{V}} \rightarrow \boxed{u^* = \frac{u}{V} \quad v^* = \frac{v}{V} \quad w^* = \frac{w}{V}}$

$u = u^* V$

$x = x^* L$ etc.

• What about $\vec{\nabla}$? → $\{\vec{\nabla}\} = \frac{1}{L}$

∴ Define $\boxed{\vec{\nabla}^* = L\vec{\nabla}}$

Cont. eq. becomes: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$$\frac{\partial(u^* V)}{\partial(x^* L)} + \frac{\partial(v^* V)}{\partial(y^* L)} + \frac{\partial(w^* V)}{\partial(z^* L)} = 0$$

Since V & L are constant, $\frac{V}{L}\left[\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right] = 0 \Rightarrow \boxed{\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0}$

$$\boxed{\vec{\nabla}^* \cdot \vec{V}^* = 0}$$

OR, STAYING IN VECTOR FORM,

$\vec{\nabla} \cdot \vec{V} = 0 \Rightarrow \left(\frac{\vec{\nabla}^*}{L}\right) \cdot (\vec{V}^* V) = 0$

OR, $\frac{V}{L}(\vec{\nabla}^* \cdot \vec{V}^*) = 0 \Rightarrow \therefore \boxed{\vec{\nabla}^* \cdot \vec{V}^* = 0}$

where

$$\boxed{\vec{\nabla}^* = \frac{\partial}{\partial x^*}\vec{i} + \frac{\partial}{\partial y^*}\vec{j} + \frac{\partial}{\partial z^*}\vec{k}}$$

Result: $\boxed{\vec{\nabla}^* \cdot \vec{V}^* = 0}$ → continuity eq. in nondimensional form

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2. Navier-Stokes Eq. · Do same thing

New, additional scaling parameters → let f = characteristic frequency
$\{f\} = \{\frac{1}{t}\}$

Let $P_0 - P_\infty$ = characteristic pressure difference $\{P_0 - P_\infty\} = \{\frac{m}{Lt^2}\}$

Pressure difference because pressure appears only as derivatives in N-S eq.

Also need g = gravitational accel.

↓

We can then Nondimensionalize the N-S eq.

Nondimensionalization of the Navier-Stokes Equation

(Section 10-2, Çengel and Cimbala)

Nondimensionalization:

We begin with the differential equation for conservation of linear momentum for a Newtonian fluid, i.e., the ***Navier-Stokes equation***. For incompressible flow,

$$\rho \frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \qquad \textbf{(10–2)}$$

Equation 10-2 is *dimensional*, and each variable or property (ρ, $\vec{V}$, t, μ, etc.) is also *dimensional*. What are the primary dimensions (in terms of {m}, {L}, {t}, {T}, etc) of each term in this equation?

$$\{\rho g\} = \left\{\frac{m}{L^3}\,\frac{L}{t^2}\right\} = \left\{\frac{m}{L^2 t^2}\right\}$$

Answer: $\left\{\frac{m}{L^2 t^2}\right\}$

To nondimensionalize Eq. 10-2, we choose ***scaling parameters*** as follows:

TABLE 10–1

Scaling parameters used to nondimensionalize the continuity and momentum equations, along with their primary dimensions

Scaling Parameter	Description	Primary Dimensions
L	Characteristic length	{L}
V	Characteristic speed	{Lt^{-1}}
f	Characteristic frequency	{t^{-1}}
$P_0 - P_\infty$	Reference pressure difference	{$mL^{-1}t^{-2}$}
g	Gravitational acceleration	{Lt^{-2}}

We define ***nondimensional variables***, using the scaling parameters in Table 10-1:

all * variables are nondimensional

$$t^* = ft \qquad \vec{x}^* = \frac{\vec{x}}{L} \qquad \vec{V}^* = \frac{\vec{V}}{V}$$

$$P^* = \frac{P - P_\infty}{P_0 - P_\infty} \qquad \vec{g}^* = \frac{\vec{g}}{g} \qquad \vec{\nabla}^* = L\vec{\nabla} \qquad \textbf{(10–3)}$$

To plug Eqs. 10-3 into Eq. 10-2, we need to first rearrange the equations in terms of the dimensional variables, i.e.,

Plug these into N-S eq.

$$t = \frac{1}{f}t^* \qquad \vec{x} = L\vec{x}^* \qquad \vec{V} = V\vec{V}^*$$

$$P = P_\infty + \left(P_0 - P_\infty\right)P^* \qquad \vec{g} = g\vec{g}^* \qquad \vec{\nabla} = \frac{1}{L}\vec{\nabla}^*$$

Now we substitute all of the above into Eq. 10-2 to obtain

$$\rho V f \frac{\partial \vec{V}^*}{\partial t^*} + \frac{\rho V^2}{L}\left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^* = -\frac{P_0 - P_\infty}{L}\vec{\nabla}^* P^* + \rho g \vec{g}^* + \frac{\mu V}{L^2}\nabla^{*2}\vec{V}^*$$

Every additive term in the above equation has primary dimensions $\{m^1L^{-2}t^{-2}\}$. To nondimensionalize the equation, we multiply every term by constant $L/(\rho V^2)$, which has primary dimensions $\{m^{-1}L^2t^2\}$, so that the dimensions cancel. After some rearrangement,

$$\boxed{\left[\frac{fL}{V}\right]}\frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^* = -\boxed{\left[\frac{P_0 - P_\infty}{\rho V^2}\right]}\vec{\nabla}^* P^* + \boxed{\left[\frac{gL}{V^2}\right]}\vec{g}^* + \boxed{\left[\frac{\mu}{\rho V L}\right]}\nabla^{*2}\vec{V}^* \qquad (10\text{–}5)$$

Strouhal number, where $\mathrm{St} = \dfrac{fL}{V}$

Euler number, where $\mathrm{Eu} = \dfrac{P_0 - P_\infty}{\rho V^2}$

Inverse of Froude number squared, where $\mathrm{Fr} = \dfrac{V}{\sqrt{gL}}$

Inverse of Reynolds number, where $\mathrm{Re} = \dfrac{\rho V L}{\mu}$

All these parameters are non dimensional

Thus, Eq. 10-5 can therefore be written as

Navier-Stokes equation in nondimensional form:

$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^* \cdot \vec{\nabla}^*)\vec{V}^* = -[\mathrm{Eu}]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^* \qquad (10\text{–}6)$$

Nondimensionalization vs. ***Normalization***:

Equation 10-6 above is *nondimensional*, but not necessarily *normalized*. What is the difference?

- ***Nondimensionalization*** concerns only the *dimensions* of the equation – we can use *any* value of scaling parameters L, V, etc., and we always end up with Eq. 10-6.
- ***Normalization*** is more restrictive than nondimensionalization. To *normalize* the equation, we must choose scaling parameters L, V, etc. that are appropriate for the flow being analyzed, such that ***all nondimensional variables*** (t^*, $\vec{V}^*$, P^*, etc.) ***in Eq. 10-6 are of order of magnitude unity***. In other words, their minimum and maximum values are reasonably close to 1.0 (e.g., $-6 < P^* < 3$, or $0 < P^* < 11$, but *not* $0 < P^* < 0.001$, or $-200 < P^* < 500$). We express the normalization as follows:

$$t^* \sim 1, \quad \vec{x}^* \sim 1, \quad \vec{V}^* \sim 1, \quad P^* \sim 1, \quad \vec{g}^* \sim 1, \quad \vec{\nabla}^* \sim 1$$

If we have properly normalized the Navier-Stokes equation, we can compare the relative importance of various terms in the equation by comparing the *relative magnitudes* of the nondimensional parameters St, Eu, Fr, and Re.

C. The Creeping Flow Approx $\equiv$ Low Reynolds # Flow

Also called Stokes Flow

$Re \ll 1$ $\qquad$ $\boxed{Re = \frac{\rho V L}{\mu}}$

$Re \ll 1$ if either μ very big (eg. honey)

or V very small (eg. glacier flow)

or L very small (eg. microorganisms, air pollution)

or some combination of the above

Consider creeping flow that is steady, incompressible w/o gravity effects

N-S eq. is $\quad (\vec{V}\cdot\vec{\nabla})\vec{V} = -\vec{\nabla}P + \mu\nabla^2\vec{V}$

nondimensional form (from above eq.)

$$\boxed{\cancel{(\vec{V}^*\cdot\vec{\nabla}^*)\vec{V}^*} = -[Eu]\,\vec{\nabla}^* P^* + \left[\frac{1}{Re}\right]\nabla^{*2}\vec{V}^*} \quad (1)$$

ignore

If properly normalized, the all circled terms are of order 1

order 1, $\sim 1 \Rightarrow$ means approx. 1. eg., $\frac{1}{2}$, -5 to 3 etc.

[Not 10^{-4} to 10^{-3} or 500 to 5000]

But, for creeping flow, $Re \ll 1$, $\therefore$ last term in (1) $\gg$ first term.

Creeping Flow Approximation $\rightarrow$ neglect the LHS of N-S eq.

$\boxed{\text{Ignore the inertial terms compared to the viscous terms}}$

Also, viscous term must be balanced by the pressure term

i.e., $\boxed{Eu \text{ must be } \sim \frac{1}{Re}}$

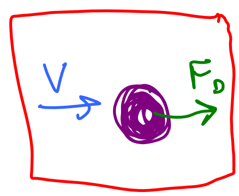
∴ Nondimen. N-S eq. becomes

$$[Eu]\,\vec{\nabla}^* p^* = \left[\frac{1}{Re}\right] \nabla^{*2} \vec{V}^*$$

back to dimensional form: $\boxed{\vec{\nabla} P \approx \mu \nabla^2 \vec{V}}$ CREEPING FLOW APPROXIMATION

Comments:

- No inertial terms!
  - there is negligible inertia in creeping flow
    microorganism cannot glide
- Density has dropped out! ρ not important in creeping flow
  except → to calc. Re — $Re = \rho \frac{VL}{\mu}$
  & if gravity effects & buoyancy are important.
- Example → Aerodynamic drag on a sphere in creeping flow



Dim. anal. → $F_D = fnc(\mu, V, L)$ → $\boxed{F_D = \text{const}\ \mu V L}$

or exact anal. → $\boxed{F_D = 3\pi \mu V D}$

(Density is not in the equation)