Today, we will:

- Continue discussing Chapter 10 – Inviscid regions of flow; irrotational regions of flow

D. Approximation for Inviscid Regions of Flow

1. Definition of Inviscid Regions of Flow and the Euler Equation

Definition: An inviscid region of flow is a region of flow in which net viscous forces are negligible compared to pressure and/or inertial forces.

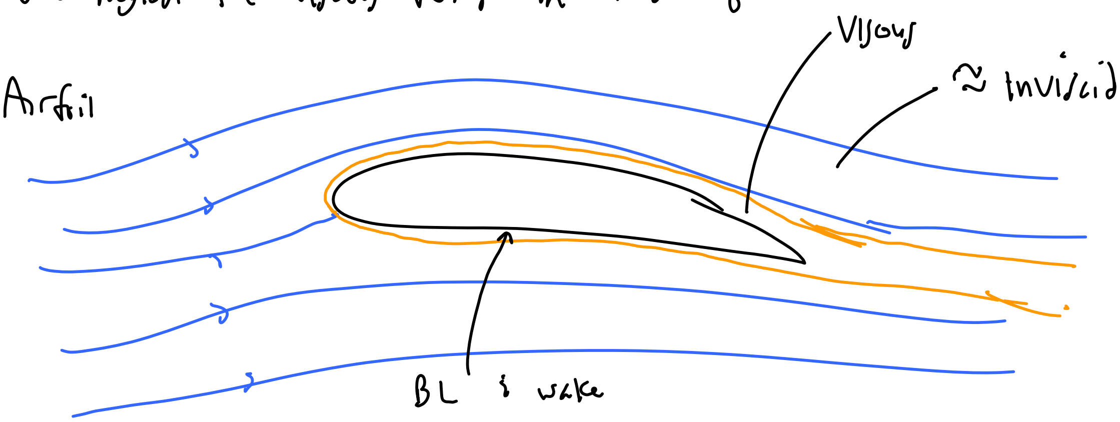
NOTE "Inviscid" does not mean that the viscosity is zero

All fluids have viscosity

But – there are regions in some flows where viscous effects are negligible compared to inertial & pressure effects

∴ We neglect the viscous terms in N-S eq.

Eg. Airfoil



Mathematically, this occurs at high Re (opposite of creeping flow)

- normalized N-S eq:

$$[St]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^*\cdot\vec{\nabla}^*)\vec{V}^* = -[Eu]\vec{\nabla}^*P^* + \left[\frac{1}{Fr}\right]\vec{g}^* + \cancel{\left[\frac{1}{Re}\right]}\nabla^{*2}\vec{V}^*$$

for Re >> 1 we ignore the viscous term

- Dimensional variables, N-S reduces to

$$\boxed{\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g}}$$

EULER EQ

Valid in inviscid regions of flow

≡ N-S eq. w/ viscous terms neglected

2. The Bernoulli Eq.

- In Ch. 5 (recall) → Bernoulli eq. is a degenerate form of the energy eq.

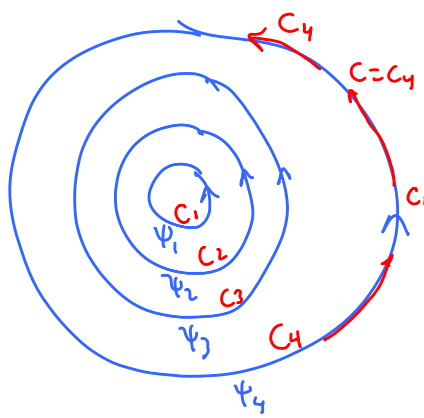
$$\boxed{\frac{P}{\rho g} + \frac{V^2}{2g} + z = \text{Constant along a streamline}} \quad ☆$$

- We can derive (see text) the same eq. for inviscid regions of flow (from the Euler eq)

- E.g. → see Eg. 10-3 in text (solid body rotation)
  ↓ no shear, ∴ viscous terms go away

∴ Euler eq. & Bernoulli eq apply

- Summary



$$\frac{P}{\rho} + \frac{V^2}{2} + gz = C$$

C varies → C is different on different streamlines

C = constant along any one streamline

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant along streamlines in inviscid regions of flow.}$$

E. The Irrotational Flow Approximation [Sec 10-5]

1. Intro recall, Vorticity = $\boxed{\vec{\zeta} = \vec{\nabla} \times \vec{V}}$ (curl of $\vec{V}$)

- If $\vec{\zeta} = 0$, flow is irrotational in some region of flow
- If $\vec{\zeta} \neq 0$, flow is rotational " "

• From math class, an identity: if $\vec{\nabla} \times \vec{V} = 0$, then $\vec{V} = \vec{\nabla}\phi$ [ϕ = potential function]

for any vector $\vec{V}$

Here, let $\vec{V} = \vec{V}$ = velocity vector. So,

if the flow is irrotational, then $\vec{\zeta} = \vec{\nabla} \times \vec{V} = 0$, $\therefore$ we can write $\vec{V}$ as $\boxed{\vec{V} = \vec{\nabla}\phi}$ ☆

$\boxed{\phi = \text{velocity potential (a scalar)}}$

Summary $\boxed{\text{If the flow is irrotational, then } \vec{V} = \vec{\nabla}\phi}$ ☆

☆ Irrotational flows are also called potential flows.

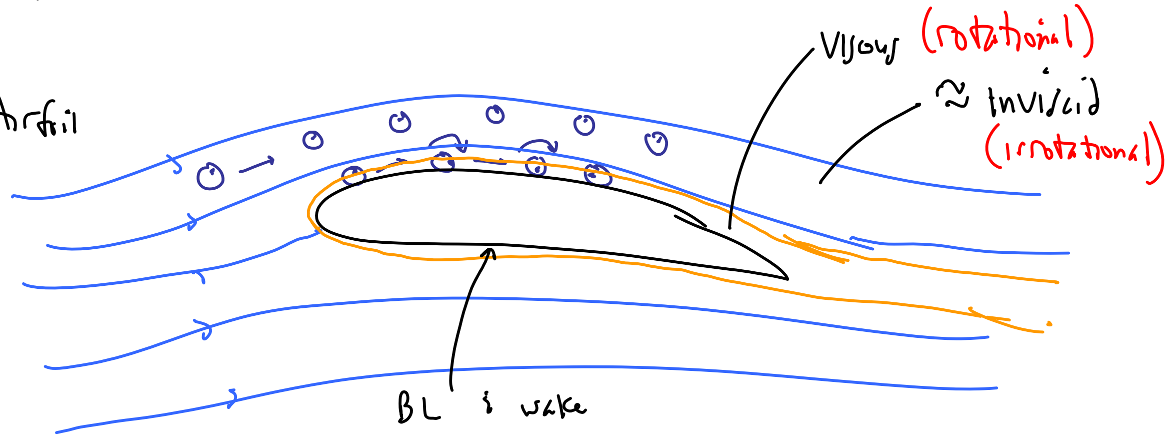
$\vec{V} = \vec{\nabla}\phi \Rightarrow$

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z} \quad \text{Cartesian coord}$$

$$u_r = \frac{\partial \phi}{\partial r} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad u_z = \frac{\partial \phi}{\partial z} \quad \text{cyl. coord.}$$

• Usefulness:

Eg. Airfoil



2. Eq's of motion for irrotational flow

a. Continuity $\vec{\nabla} \cdot \vec{V} = 0$ but $\boxed{\vec{V} = \vec{\nabla}\phi}$

$\therefore \vec{\nabla} \cdot (\vec{\nabla}\phi) = 0 \Rightarrow \boxed{\nabla^2 \phi = 0}$ = Laplacian eq. ☆

b. N-S eq

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \underbrace{\mu \nabla^2 \vec{V}}_{\mu \nabla^2(\vec{\nabla}\phi)} \qquad \vec{V} = \vec{\nabla}\phi$$

But, for a smooth continuous function ϕ, the order of differentiation does not matter.

$$\cancel{\mu \vec{\nabla}(\nabla^2 \phi)} = 0 \qquad \nabla^2\phi = 0 \text{ (cont)}$$

N-S eq reduces to the Euler Eq.

$$\boxed{\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g}}$$

Euler Eq holds for irrotational flow ☆

c. Bernoulli eq. (See text for derivation)

When $\vec{\zeta} = 0$ (irrotational region of flow) then

☆ ☆

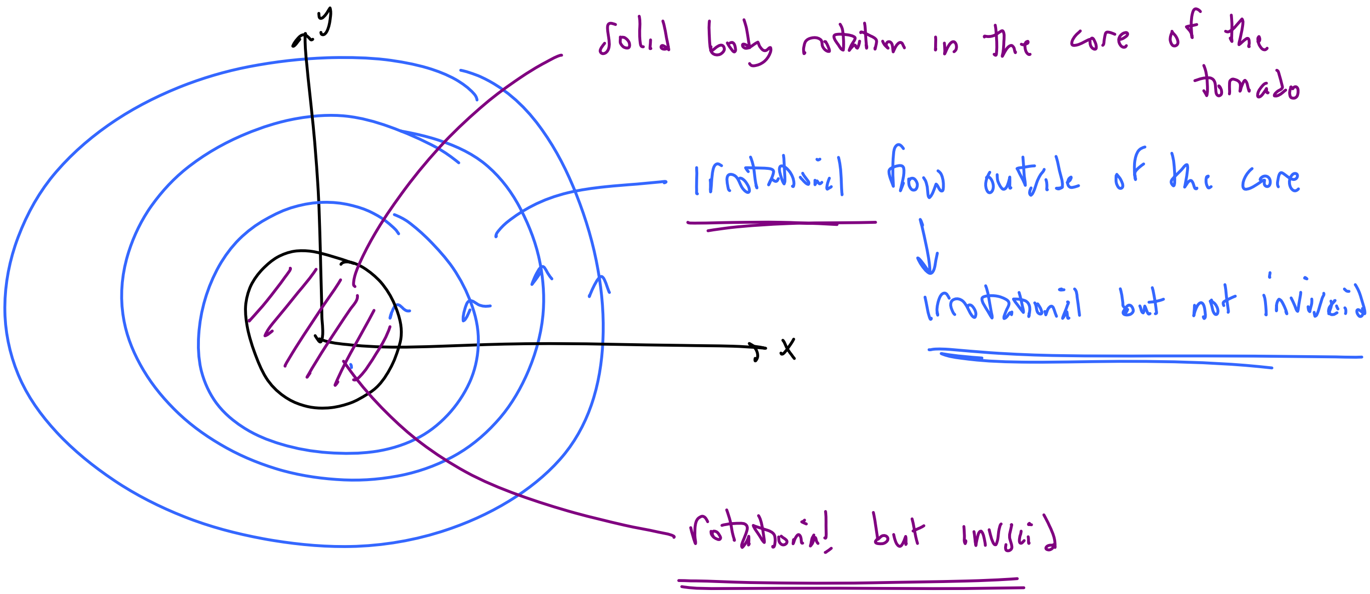
$$\boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = C = \text{constant everywhere in the irrotational region}}$$

Compare

- Inviscid region → Bernoulli eq is valid, but C is const. only along streamlines
- Irrotational region → " " ", C is const. everywhere.

the irrotational flow approx. is more restrictive than the inviscid approx

d. Examples See Eg 10-4 in text → Simple model of a tornado



3. 2-D Irrotational Flow

a. **Approximations:**
- 2-D
- Steady
- incompressible
- irrotational

Eqs:

$$\boxed{\vec{V} = \vec{\nabla}\phi} \Rightarrow \left[u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \right]$$

- **Continuity** $\vec{\nabla}\cdot\vec{V} = 0 \Rightarrow \boxed{\nabla^2 \phi = 0}$ Laplace eq.

$$\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \right]$$

- **Stream function**

$$\boxed{u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}}$$

- Irrotationality $\vec{\zeta} = 0 \rightarrow \vec{\nabla}\times\vec{V} = 0$ $\quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

In 2-D, only the z-component remains

Plug in ψ: $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$$-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0 \Rightarrow \boxed{\nabla^2 \psi = 0}$$

Stream fnc. also satisfies Laplace Eq!

- N-S eq → reduces to Euler eq.

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} p + \rho \vec{g}$$, which reduces to Bernoulli's eq

$$\boxed{\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant everywhere}}$$

SUMMARY OF EQ's FOR 2D, steady, incompressible, irrotational flow:

$$\boxed{\vec{V} = \vec{\nabla}\phi} \quad \boxed{\nabla^2 \phi = 0} \quad \boxed{u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}} \quad \boxed{\nabla^2 \psi = 0}$$

$$\boxed{\vec{\zeta} = \vec{\nabla} \times \vec{V} = 0 \quad \text{(irrotational)}}$$

&

$$\boxed{\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant everywhere}}$$

[or, in terms of head, $\boxed{\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{constant everywhere}}$]