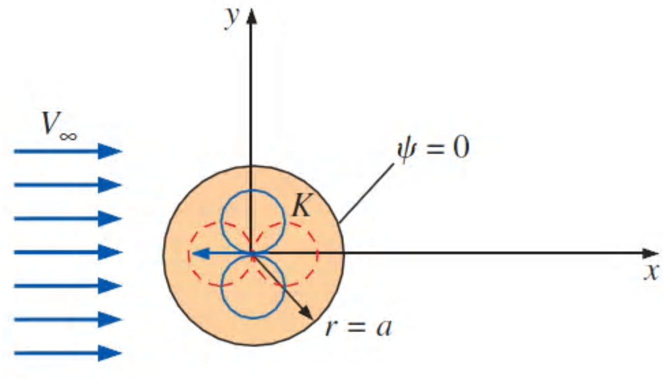
**Today, we will**:

A "classic"

- Do another example of superposition of irrotational flows – flow over a circular cylinder
- Start discussing the last approximation of Chapter 10: **The Boundary Layer Approx.**

## b. Example of superposition: Flow over a circular cylinder



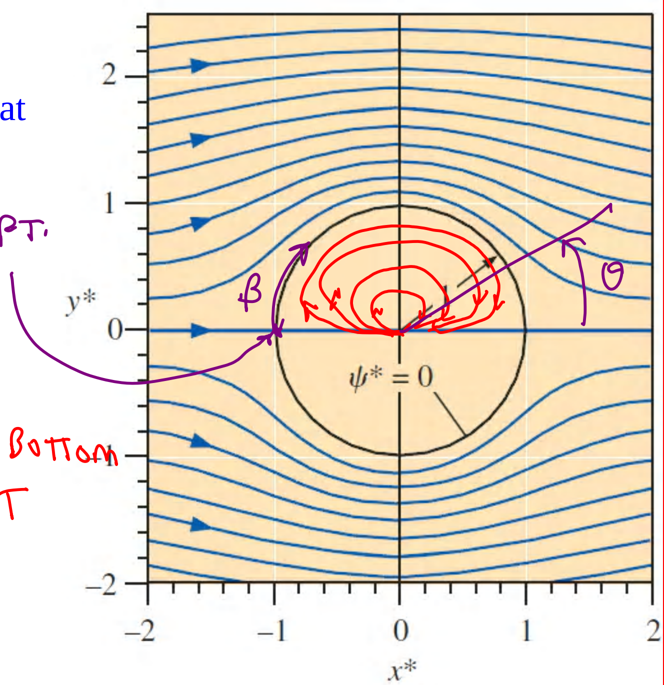

**Given**: Superpose a uniform stream of velocity $V_\infty$ and a doublet of strength $K$ at the origin.

**To do**: Plot streamlines, and discuss the flow that results from this superposition.

**Solution**:

- We simply add up the stream functions for the two building block flows: $\psi = \psi_{\text{freestream}} + \psi_{\text{doublet}} = V_\infty y - K\dfrac{\sin\theta}{r}$ .
- But we know that $\boxed{y = r\sin\theta}$ , thus, $\boxed{\psi = V_\infty r\sin\theta - K\dfrac{\sin\theta}{r}}$.
- For "convenience", and with hindsight, we choose to set $\psi = 0$ at $r = a$.
  [It turns out that radius $a$ is a special radius that becomes the radius of the circle.]
- Set $r = a$ in our equation for the stream function:
  $$0 = V_\infty a\sin\theta - K\frac{\sin\theta}{a} \quad \rightarrow \quad \boxed{K = V_\infty a^2}.$$
- Then our final expression for $\psi$ becomes
  $$\boxed{\psi = V_\infty \sin\theta\left(r - \frac{a^2}{r}\right)}.$$

SYMMETRIC TOP & BOTTOM AND FORE & AFT

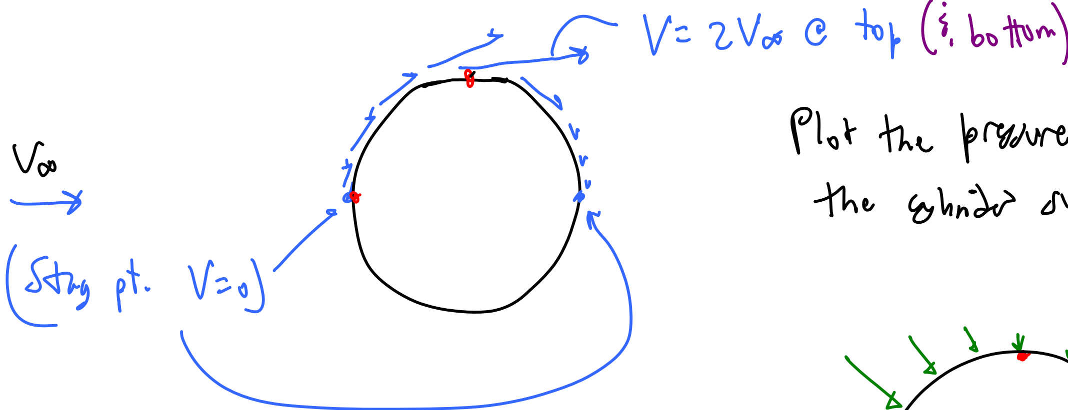



- Plot streamlines: [we plot nondimensionally, setting $x^* = x/a$ and $y^* = y/a$]
- From our equation for $\psi$ above, we can calculate the velocity field from the definition of $\psi$, i.e., $u_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \quad u_\theta = -\dfrac{\partial\psi}{\partial r}$ . See text for details. On the cylinder ($r = a$),
  $\boxed{u_r = 0 \quad u_\theta = -2V_\infty\sin\theta}$

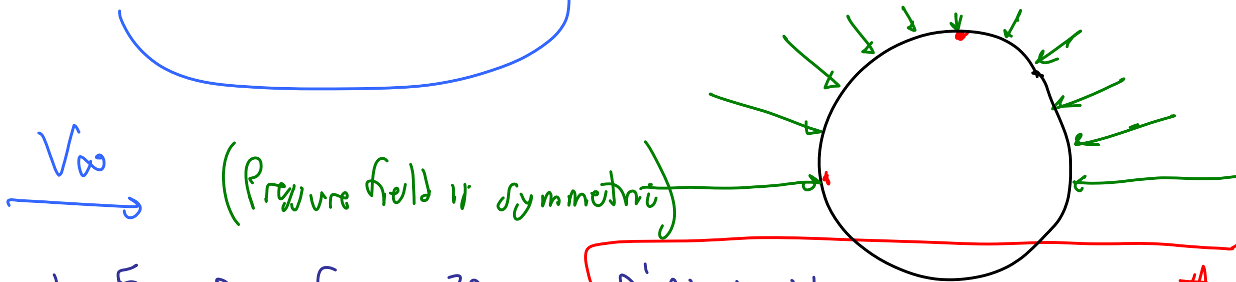
(Use Bernoulli to calculate the pressure)

- We can also define the **pressure coefficient**, $\boxed{C_p = \dfrac{P - P_\infty}{\frac{1}{2}\rho V_\infty^{\,2}} = 1 - \dfrac{V^2}{V_\infty^{\,2}}}$.
- *On the cylinder*, it turns out that $\boxed{C_p = 1 - 4\sin^2\beta}$, where $\beta$ is the angle from the nose.

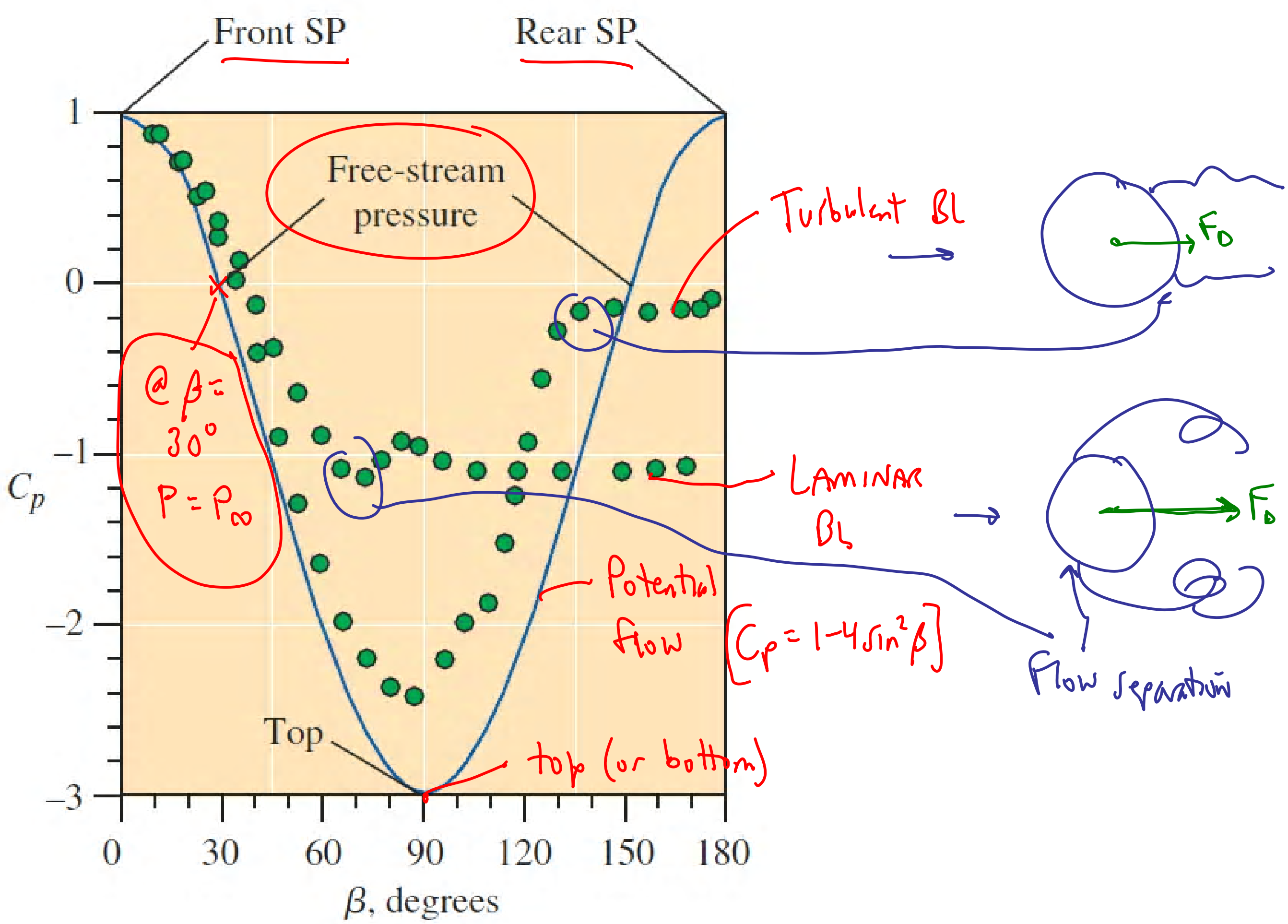
· Plot velocity vectors along the cyl. surface:




Plot the pressure field on the cylinder surface:




We get $F_D$ = Drag force = $\underline{\text{zero}}$ → D'Alembert's paradox
Drag = 0 for any non-lifting body in irrotational flow



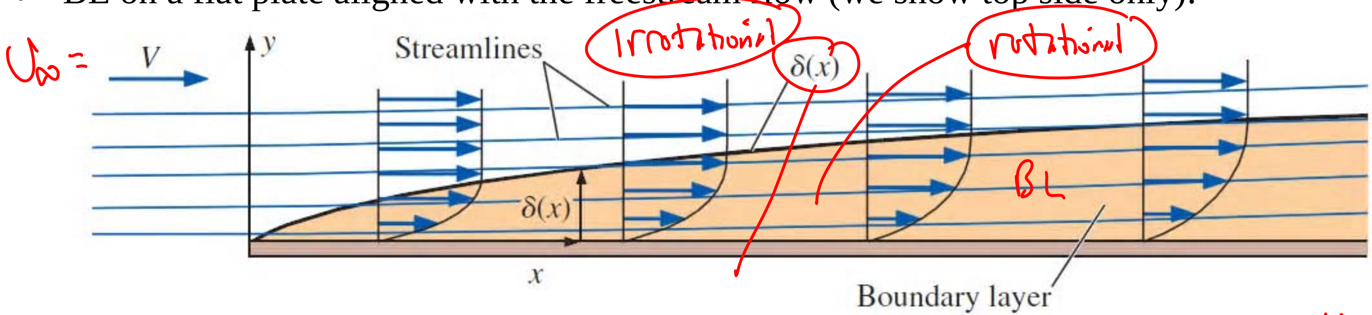
## F. The Boundary Layer Approximation

1. Introduction

Definition: A ***boundary layer*** is a thin layer in which viscous effects and vorticity are significant, and cannot be ignored.

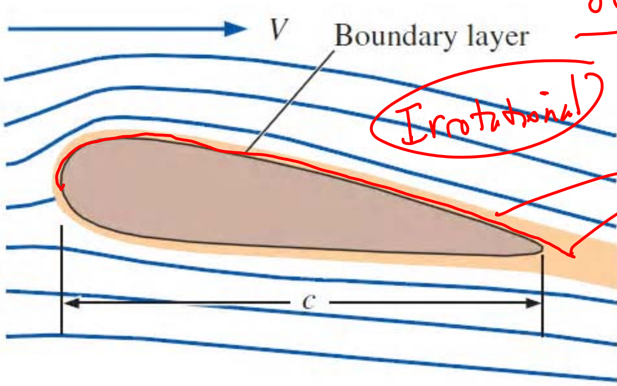
Examples

- BL on a flat plate aligned with the freestream flow (we show top side only):




- BL on an airfoil:

$\delta(x)$ = BL thickness, typically where $u = 0.99\,U_\infty$



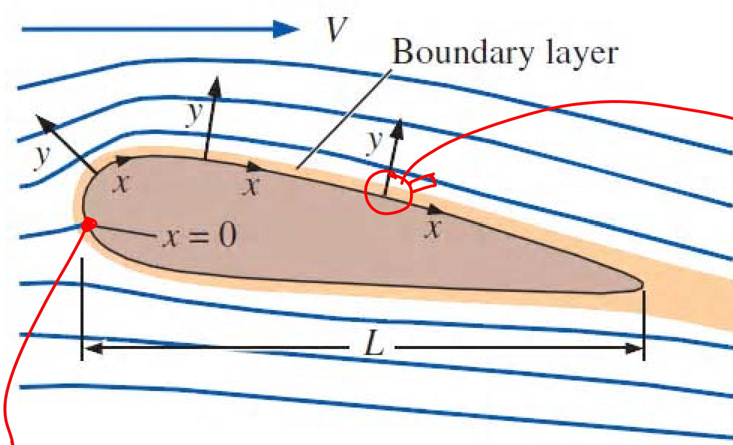

These BLs are exaggerated. They are in real life much thinner

Comments:
- N-S eq is valid everywhere, but too hard to solve.
- Euler eq or irrotational approx. is easy, but does not apply near walls.
- The BL approx "bridges the gap" between N-S & Euler eqs. (useful near walls & in wakes)

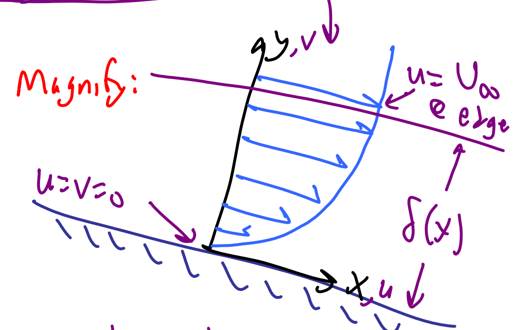
2. The Boundary Layer Coordinate System

In a 2-D flow, we let *x* = distance along the wall, and *y* = distance normal to the wall.

local coord. system




Magnify:

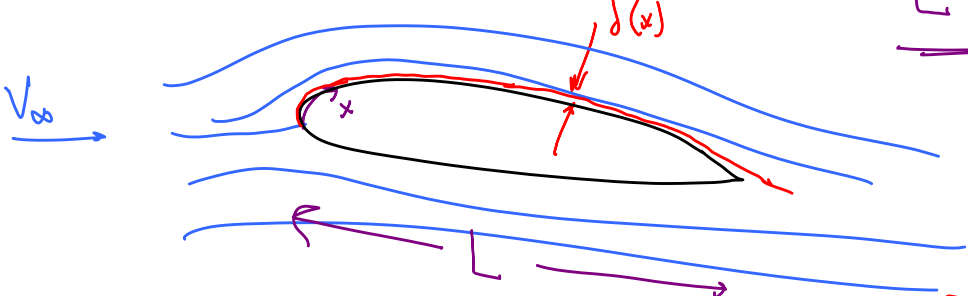



x starts @ the stag. pt. @ front of body

* no-slip condition (u=v=0 @ wall)

## 3. The BL Eqs

- Nondimensionalize the continuity & N-S eqs as previously
- Here we have 2 length scales instead of just 1 (previously) → L & $\delta$




- The BL Approximation is that $\boxed{\delta \ll L}$ ☆☆☆

☆ i.e. the BL is very THIN

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Nondimensionalization → previously $\frac{\partial}{\partial x} \sim \frac{1}{L}$ $\quad \frac{\partial}{\partial y} \sim \frac{1}{L}$

Here for a BL, $\boxed{\frac{\partial}{\partial x} \sim \frac{1}{L} \quad \text{but} \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta}}$ ☆

more appropriate since BL is so thin

- Continuity (2-D, steady, incomp., ignore gravity)

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

$\boxed{u \gg v}$ ☆ but $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$ } Both terms remain

- x-mom (N-S)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\cancel{\frac{\partial^2 u}{\partial x^2}} + \nu\frac{\partial^2 u}{\partial y^2}$$

$\approx 0$

$\sim \nu\frac{u}{L^2}$ $\quad \sim \nu\frac{u}{\delta^2}$

but $\delta \ll L$, the last term is way bigger than the 2nd-to-last term

BL approx. → x-mom eq.

(Valid inside the BL)

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}}$$

• y-mom (N-S eq)

negligible

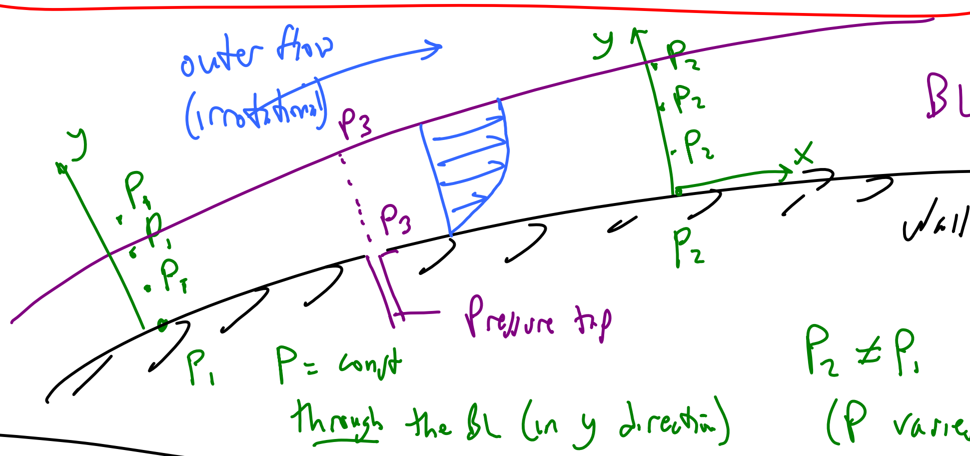
$$\cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \cancel{\nu\frac{\partial^2 v}{\partial x^2}} + \cancel{\nu\frac{\partial^2 v}{\partial y^2}}$$

See text for details → all terms are small compared to pressure term

y-mom eq. w/ BL approx reduces to $\boxed{\frac{\partial P}{\partial y} \simeq 0}$ ☆ in a BL

Interpretation of $\frac{\partial P}{\partial y} = 0$ in a BL:

P does not vary across a BL, only along the BL. A




$P_2 \neq P_1$

(P varies along the BL (x-dir))

$$\boxed{\frac{\partial P}{\partial y} \simeq 0, \text{ but } \frac{\partial P}{\partial x} \neq 0 \text{ in a BL in general}}$$ ☆

Pressure can be measured at the wall & it is the same pressure at the outer edge of the BL

↙

Significant in the design of airfoils → can use potential flow to predict the lift on an airfoil & test in a wind tunnel - measure P

## 4. The BL Procedure (5 steps)

- **Step 1** Solve for the outer flow (irrotational) $U(x)$
  $P(x)$

- **Step 2** Assume a thin BL (so thin that $U(x)$ is not affected)

- **Step 3** Solve the BL eqs

- **Step 4** Calculate quantities of interest (drag, etc.)

- **Step 5** Verify the BL is indeed thin
  (compare $\delta$ & $L$)
  $\downarrow$
  verify that $\delta \ll L$

We will do some examples next time using this BL procedure