Today, we will:

- Discuss the turbulent boundary layer on a flat plate, and compare with laminar flow
- Talk about boundary layers with pressure gradients

F. The Boundary Layer Approximation (continued)

6. Turbulent BL on a Flat Plate

At high enough Re (Re_x), the BL will be turbulent.

\[ \tau_w = \frac{1}{2} \left( \frac{d\tau}{dy}\right)_w \]

Turb. BL is more “full” than a laminar BL under the same condition.

-Similar to pipe flow:

Turb. BL has larger shear stress than a laminar BL under the same condition (same Re).
The turbulent flat plate boundary layer velocity profile:

The time-averaged turbulent flat plate (zero pressure gradient) boundary layer velocity profile is much **fuller** than the laminar flat plate boundary layer profile, and therefore has a larger slope $\partial u / \partial y$ at the wall, leading to greater skin friction drag along the wall.

There are three common empirical relationships for the turbulent flat plate boundary layer velocity profile:

- **The log law:**

  \[
  \frac{u}{u_\infty} = \frac{1}{\kappa} \ln \frac{y u_\infty}{\nu} + B
  \]  

  where

  \[
  u_\infty = \sqrt{\frac{\tau_w}{\rho}}
  \]  

  \[
  \tau_w = \mu \frac{\partial u}{\partial y}
  \]

  at $y = 0$

We define the engineering critical Reynolds number:

- $Re_x < Re_{x,cr}$ BL most likely laminar
- $Re_x > Re_{x,cr}$ BL most likely turbulent

Note: The vertical scale is exaggerated for clarity.
- Spalding’s law of the wall:
\[
\frac{y u_s}{v} = \frac{u}{u_s} + e^{-\kappa B} \left[ e^{\kappa (u/u_s)} - 1 - \kappa (u/u_s) + \frac{[\kappa (u/u_s)]^2}{2} - \frac{[\kappa (u/u_s)]^3}{6} \right]
\] (10–85)

- The one-seventh-power law:
\[
\frac{u}{U} \approx \left( \frac{y}{\delta} \right)^{1/7} \quad \text{for } y \leq \delta, \quad \Rightarrow \quad \frac{u}{U} \approx 1 \quad \text{for } y > \delta \] (10–82)

**Quantities of interest for the turbulent flat plate boundary layer:**

Just as we did for the laminar (Blasius) flat plate boundary layer, we can use these expressions for the velocity profile to estimate quantities of interest, such as the 99% boundary layer thickness \( \delta \), the displacement thickness \( \delta^* \), the local skin friction coefficient \( C_{f,x} \), etc. These are summarized in Table 10-4 in the text.

**Table 10-4**

<table>
<thead>
<tr>
<th>Property</th>
<th>Laminar</th>
<th>Turbulent(^{(1)})</th>
<th>Turbulent(^{(2)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary layer thickness</td>
<td>( \frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}} )</td>
<td>( \frac{\delta}{x} \approx \frac{0.16}{(Re_x)^{1/7}} )</td>
<td>( \frac{\delta}{x} \approx \frac{0.38}{(Re_x)^{1/5}} )</td>
</tr>
<tr>
<td>Displacement thickness</td>
<td>( \frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}} )</td>
<td>( \frac{\delta^*}{x} \approx \frac{0.020}{(Re_x)^{1/7}} )</td>
<td>( \frac{\delta^*}{x} \approx \frac{0.048}{(Re_x)^{1/5}} )</td>
</tr>
<tr>
<td>Momentum thickness</td>
<td>( \frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}} )</td>
<td>( \frac{\theta}{x} \approx \frac{0.016}{(Re_x)^{1/7}} )</td>
<td>( \frac{\theta}{x} \approx \frac{0.037}{(Re_x)^{1/5}} )</td>
</tr>
<tr>
<td>Local skin friction coefficient</td>
<td>( C_{f,x} = \frac{0.664}{\sqrt{Re_x}} )</td>
<td>( C_{f,x} \approx \frac{0.027}{(Re_x)^{1/7}} )</td>
<td>( C_{f,x} \approx \frac{0.059}{(Re_x)^{1/5}} )</td>
</tr>
</tbody>
</table>

Note that \( C_{f,x} \) is the *local* skin friction coefficient, applied at only one value of \( x \).

To these we add the integrated *average skin friction coefficients* for one side of a flat plate of length \( L \), noting that \( C_f \) applies to the entire plate from \( x = 0 \) to \( x = L \) (see Chapter 11):

**Laminar:** \( C_f = \frac{1.33}{Re_L^{1/2}} \) \( \Re_L \leq 5 \times 10^5 \)

**Turbulent:** \( C_f = \frac{0.074}{Re_L^{1/5}} \) \( 5 \times 10^5 \leq \Re_L \leq 10^7 \)

\( C_f, \delta, \delta^*, \text{etc. (11–19)} \) are all larger for turb. BL than for a laminar BL.
For cases in which the laminar portion of the plate is taken into consideration, we use:

\[
C_f = \frac{0.074}{\text{Re}_{L}^{1/5}} - \frac{1742}{\text{Re}_{L}} \quad 5 \times 10^5 \leq \text{Re}_{L} \leq 10^7
\]  

(11-22)

**Turbulent flat plate boundary layers with wall roughness:**

Finally, all of the above are for smooth flat plates. However, if the plate is rough, the average skin friction coefficient \(C_f\) increases with roughness \(\varepsilon\). This is similar to the situation in pipe flows, in which Darcy friction factor \(f\) increases with pipe wall roughness.

![Graph showing friction coefficient for parallel flow over smooth and rough flat plates.](image)

**FIGURE 11–31**

Friction coefficient for parallel flow over smooth and rough flat plates.

Just as with pipe flows, at high enough Reynolds numbers, the boundary layer becomes “fully rough”. For a fully rough flat plate turbulent boundary layer with average wall roughness height \(\varepsilon\),

**Fully rough turbulent regime:**

\[
C_f = \left(1.89 - 1.62 \log \frac{\varepsilon}{L}\right)^{-2.5}
\]

(11-23)

This equation represents the flat portions of Fig. 11-31 that are labeled “Fully rough”.

Increasing wall roughness
Example

\[ V = 51.3 \text{ ft/s} \]

\[ L = 8' \]

\[ \delta = \frac{0.764 \text{ ft}}{0.003883} \approx 0.76 \text{ lbf} \]

\[ \text{Re}_x = \frac{UL}{\nu} = 2.51 \times 10^6 \]

\[ \delta = 0.297 \text{ in} \text{ at } x = L \]

\[ F_D = 0.17 \text{ lbf} \text{ (laminar)} \]

\[ \text{If } \delta \ll L, \text{ BL is thin laminar} \]

\[ \frac{\delta}{x} = \frac{0.38}{(\text{Re}_x)^{1/5}} \rightarrow \delta = 0.1595 \text{ ft} \]

\[ \delta = 1.9 \text{ inches} \]

\[ \text{BL is } 6.5 \times \text{ thicker than laminar BL at same Re} \]

Calc. \( F_D \) on plate

\[ F_D = \frac{1}{2} \rho V^2 C_f A \]

From table, \( C_f = \frac{0.074}{\text{Re}_L^{1/5}} \)

\[ C_f = 0.003883 \]

\[ F_D = 0.764 \approx 0.76 \text{ lbf} \]

\[ \text{Actual value of } F_D \text{ will be larger than this} \]
Re-do if plate is rough with \( \varepsilon = 0.05 \) in
\[
\frac{L}{\varepsilon} = \frac{96''}{0.05''} = 1920
\]
\[
C = \frac{L}{\varepsilon} = 1920 \quad \Rightarrow \quad Re = 2.5 \times 10^6, \quad C_f = \frac{\tau_w}{\frac{v^2}{2}}, \quad \frac{F_0}{1.377} = 1.4 \text{ lbf} = F_0
\]

\( (= 2 \times \text{as smooth case}) \)

7. **BL's with pressure gradient**

a. Definition → For a flat plate BL
\[
U(x) = U = V = \text{cont.} \quad P(x) = P = \text{cont. everywhere}
\]

* Zero pressure gradient BL

In reality, most BL's have non-zero pressure gradient

i.e., \( P(x) \neq \text{cont.} \rightarrow P \text{ changes with } x \)

e.g. Airfoil \( \alpha < \) of attack

\[
U(x) \uparrow \text{with } x \quad \Rightarrow \quad \frac{\partial P}{\partial x} > 0 \quad \text{Adverse or unfavorable pressure gradient}
\]

\[
U(x) \downarrow \text{with } x \quad \Rightarrow \quad \frac{\partial P}{\partial x} < 0 \quad \text{Favorable pressure gradient}
\]

\[
P(x) \uparrow \text{with } x \quad \Rightarrow \quad \text{Boundary layer more likely to separate}
\]
BL separation for an airfoil at high $\alpha$ of attack

Separation Point

STALL

Example - a diffuser

nice shallow diffuser

$\downarrow$ $P$ invariant

sharp angle diffuser

flow separation

Contraction have a favorable pressure gradient ($P$ decreasing)

so they are much less likely to separate

$P(x)$ is decreasing
My wind tunnel in Fearby Lab:

*contraction is very steep*

*Shallow angle diffuser*

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Favorable pressure gradient

\[ U \uparrow, P \downarrow \]

: can make it short

(it will not separate)

Adverse pressure gradient

\[ U \downarrow, P \uparrow \]

: must make the diffuser sections long to avoid separation

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Similar for Venturi meters (Ch. 8):

*Sharp contraction* (favorable p.g.)

*long diffuser* (adverse p.g.)