Today, we will:

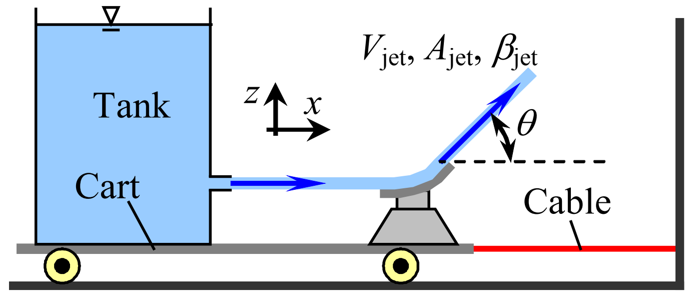
- Do some more example problems – linear CV momentum equation
- Discuss the control volume equation for angular momentum

E. The Linear Momentum Equation for a Control Volume (continued)

4. Examples (continued)

Example: Tension in a cable

Given: A cart with frictionless wheels and a large tank shoots water at a deflector plate, turning it by angle θ as sketched. The cart tries to move to the left, but a cable prevents it from doing so. At the exit of the deflector, the water jet area A_{jet}, its average velocity V_{jet}, and its momentum flux correction factor β_{jet} are known.



To do: Calculate the tension T in the cable.

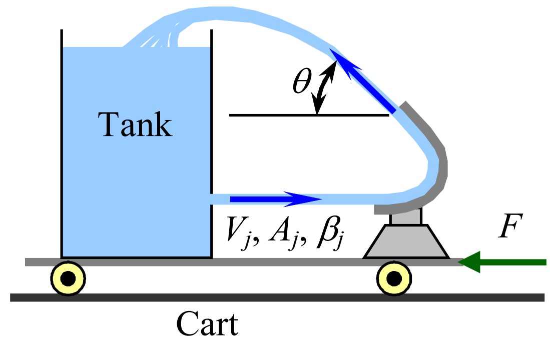
Solution:

- First step:
- Second step: Use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

Example: Force to hold a cart in place

Given: Water shoots out of a large tank sitting on a cart. The water jet velocity is $V_j = 7.00$ m/s, its cross-sectional area is $A_j = 20.0$ mm^2, and the momentum flux correction factor of the jet is 1.04. The water is deflected 135° as shown ($\theta = 45°$), and all of the water flows back into the tank. The density of the water is 1000 kg/m^3.



To do: Neglecting friction on the wheels, calculate the horizontal force F (in units of N) required to hold the cart in place.

Solution:

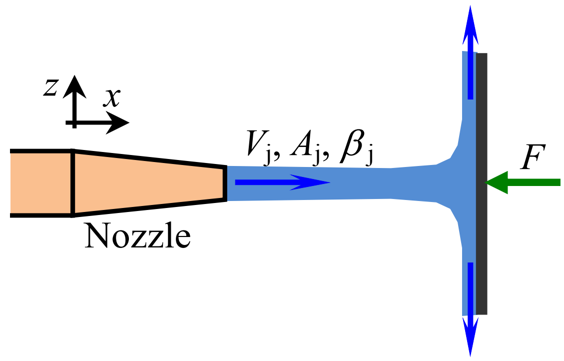
- First step:
- Second step: Use the approximate, most useful form of the linear momentum equation,

$$\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

Example: Force imparted by a water jet hitting a stationary plate

Given: A horizontal water jet of area A_j, average velocity V_j, and momentum flux correction factor β_j impinges normal to a stationary vertical flat plate.

To do: Calculate the horizontal force F required to keep the plate from moving.



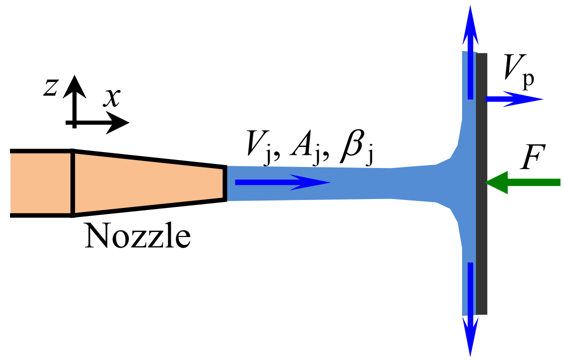
Solution:

- First step:
- Second step: Use the approximate, most useful form of the linear momentum equation,

$$\boxed{\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}}$$

Example: Force imparted by a water jet hitting a *moving* plate

Given: A horizontal water jet of area A_j, average velocity V_j, and momentum flux correction factor β_j impinges normal to a *moving* vertical flat plate. The plate moves to the right at constant speed V_p.



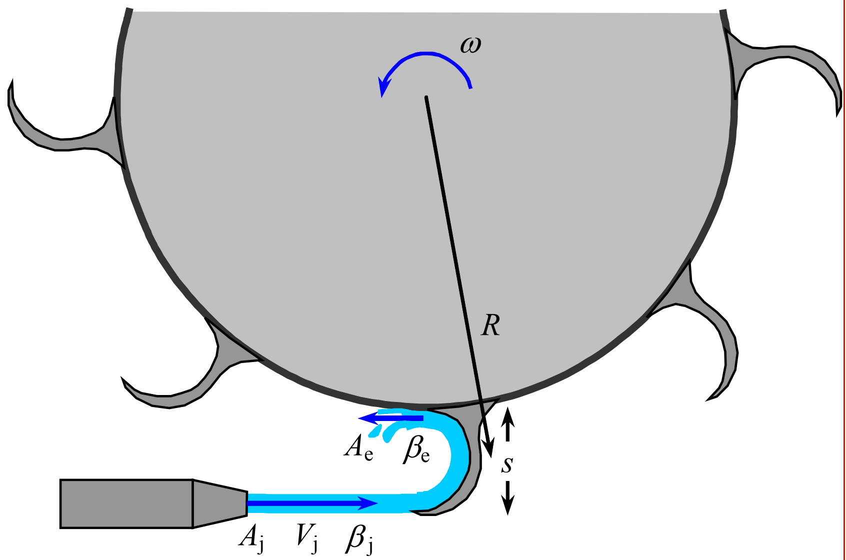
To do: Calculate the horizontal force F required to keep the plate moving at constant speed V_p.

Solution:

- First step:
- Second step: Use the approximate, most useful form of the linear momentum equation, in the x-direction, for a moving CV, but *steady*:

$$\boxed{\sum F_x = \sum F_{x,\text{ gravity}} + \sum F_{x,\text{ pressure}} + \sum F_{x,\text{ viscous}} + \sum F_{x,\text{ other}} = \sum_{\text{out}} \beta \dot{m} u_r - \sum_{\text{in}} \beta \dot{m} u_r}$$

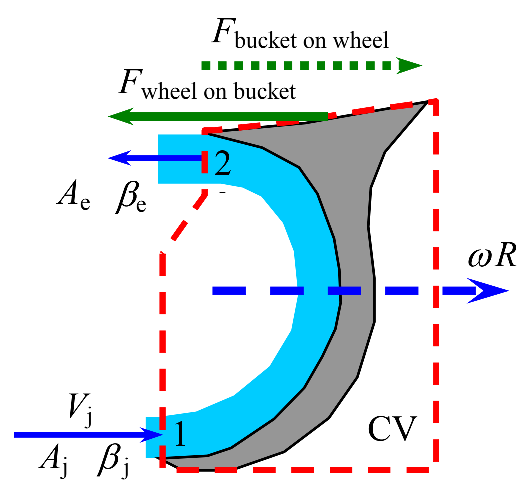
Example: Force on a bucket of a Pelton-type (impulse) hydroturbine



Given: An impulse turbine is driven by a high-speed water jet (average jet velocity V_j over jet area A_j, with momentum flux correction factor β_j) that impinges on turning buckets attached to a turbine wheel as shown. The turbine wheel rotates at angular velocity ω, and is horizontal; therefore, gravity effects are not important in this problem. (The view in the sketch is from the top.) The turning buckets turn the water approximately 180 degrees, and the water exits the bucket over exit cross-sectional area A_e with exit momentum flux correction factor β_e. For simplicity, we approximate that the bucket dimension s is much smaller than turbine wheel radius R ($s \ll R$).

(**a**) **To do**: Calculate the force of the bucket on the turbine wheel, $F_{\text{bucket on wheel}}$, at the instant in time when the bucket is in the position shown.

(**b**) **To do**: Calculate the power delivered to the turbine wheel.



Solution: We choose a control volume surrounding the bucket, cutting through the water jet at the inlet to the bucket, and cutting through the water exiting the bucket. Note that this is a *moving control volume*, moving to the right at speed ωR. We also cut through the welded joint between the bucket and the turbine wheel, where the force $F_{\text{bucket on wheel}}$ is to be calculated. Because of Newton's third law, the force acting *on the control volume* at this location is equal in magnitude, but opposite in direction, and we call it $F_{\text{wheel on bucket}}$.

Since the pressure through an incompressible jet exposed to atmospheric air is equal to P_{atm}, the pressure at the inlet (1) is equal to P_{atm}, and the pressure at the exit (2) is also equal to P_{atm}.

Solution to be completed in class.

- Use the x-component of the steady linear momentum equation for a moving CV,

$$\boxed{\sum F_x = \sum F_{x,\text{ gravity}} + \sum F_{x,\text{ pressure}} + \sum F_{x,\text{ viscous}} + \sum F_{x,\text{ other}} = \sum_{\text{out}} \beta \dot{m} u_r - \sum_{\text{in}} \beta \dot{m} u_r}$$

Angular Momentum Control Volume Analysis
(Section 6-6, Çengel and Cimbala)

1. <u>Equations and definitions</u>

See the derivation in the book, using the Reynolds transport theorem. The result is:

General:
$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n}) \, dA \qquad \text{(6–47)}$$

(Relative velocity)

which is stated in words as

$$\begin{pmatrix}\text{The sum of all} \\ \text{external moments} \\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change} \\ \text{of the angular momentum} \\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of} \\ \text{angular momentum} \\ \text{out of the control} \\ \text{surface by mass flow}\end{pmatrix}$$

We simplify the control surface integral for cases in which there are well-defined inlets and outlets, just as we did previously for mass, energy, and momentum. The result is:

$$\sum \vec{M} \cong \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, dV + \sum_{\text{out}} (\vec{r} \times \dot{m}\vec{V}) - \sum_{\text{in}} (\vec{r} \times \dot{m}\vec{V}) \qquad \text{(6–50)}$$

Note that we cannot define an "angular momentum flux correction factor" like we did previously for the kinetic energy and momentum flux terms. Furthermore, many problems we consider in this course are *steady*. For steady flow, Eq. 6-50 reduces to:

Steady flow:
$$\sum \vec{M} = \sum_{\text{out}} (\vec{r} \times \dot{m}\vec{V}) - \sum_{\text{in}} (\vec{r} \times \dot{m}\vec{V}) \qquad \text{(6–51)}$$

Net moment or torque acting on the control volume by external means = Rate of flow of angular momentum *out* of the control volume by mass flow − Rate of flow of angular momentum *into* the control volume by mass flow

Finally, in many cases, we are concerned about only *one* axis of rotation, and we simplify Eq. 6-51 to a scalar equation,

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V \qquad \text{(6–52)}$$

Equation 6-52 is the form of the angular momentum control volume equation that we will most often use, noting that r is the shortest distance (i.e., the *normal* distance) between the point about which moments are taken and the *line of action* of the force or velocity being considered. By convention, *counterclockwise moments are positive*.

2. Examples

See Examples 6-8 and 6-9 in the book. Example 6-8 is discussed in more detail here.

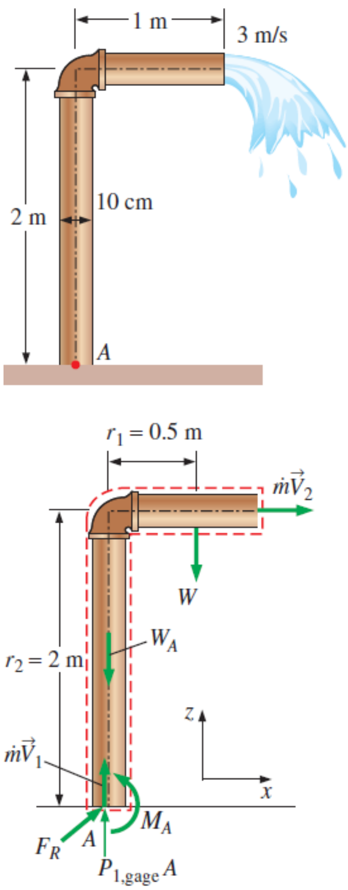


FIGURE 6–39

Schematic for Example 6–8 and the free-body diagram.

EXAMPLE 6–8 Bending Moment Acting at the Base of a Water Pipe

Underground water is pumped through a 10-cm-diameter pipe that consists of a 2-m-long vertical and 1-m-long horizontal section, as shown in Fig. 6–39. Water discharges to atmospheric air at an average velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the bending moment acting at the base of the pipe (point *A*) and the required length of the horizontal section that would make the moment at point *A* zero.

SOLUTION Water is pumped through a piping section. The moment acting at the base and the required length of the horizontal section to make this moment zero is to be determined.

Assumptions **1** The flow is steady. **2** The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero. **3** The pipe diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

Properties We take the density of water to be 1000 kg/m^3.

Analysis We take the entire L-shaped pipe as the control volume, and designate the inlet by 1 and the outlet by 2. We also take the *x*- and *z*-coordinates as shown. The control volume and the reference frame are fixed.

The conservation of mass equation for this one-inlet, one-outlet, steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}$, and $V_1 = V_2 = V$ since A_c = constant. The mass flow rate and the weight of the horizontal section of the pipe are

$$\dot{m} = \rho A_c V = (1000 \text{ kg/m}^3)[\pi(0.10 \text{ m})^2/4](3 \text{ m/s}) = 23.56 \text{ kg/s}$$

$$W = mg = (12 \text{ kg/m})(1 \text{ m})(9.81 \text{ m/s}^2)\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2}\right) = 117.7 \text{ N}$$

To determine the moment acting on the pipe at point *A*, we need to take the moment of all forces and momentum flows about that point. This is a steady-flow problem, and all forces and momentum flows are in the same plane. Therefore, the angular momentum equation in this case is expressed as

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$$

where *r* is the average moment arm, *V* is the average speed, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

These moments are moments *acting on* the CV.

These moments are moments *due to angular momentum.*

The free-body diagram of the L-shaped pipe is given in Fig. 6–39. Noting that the moments of all forces and momentum flows passing through point *A* are zero, the only force that yields a moment about point *A* is the weight *W* of the horizontal pipe section, and the only momentum flow that yields a moment is the outlet stream (both are negative since both moments are in the clockwise direction). Then the angular momentum equation about point *A* becomes

$$M_A - r_1 W = -r_2 \dot{m} V_2$$

Solving for M_A and substituting give

$$M_A = r_1 W - r_2 \dot{m} V_2$$

$$= (0.5\text{ m})(118\text{ N}) - (2\text{ m})(23.56\text{ kg/s})(3\text{ m/s})\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right)$$

$$= \mathbf{-82.5\ N\cdot m}$$

The negative sign indicates that the assumed direction for M_A is wrong and should be reversed. Therefore, a moment of 82.5 N·m acts at the stem of the pipe in the clockwise direction. That is, the concrete base must apply a 82.5 N·m moment on the pipe stem in the clockwise direction to counteract the excess moment caused by the exit stream.

The weight of the horizontal pipe is $w = W/L = 117.7$ N per m length. Therefore, the weight for a length of Lm is Lw with a moment arm of $r_1 = L/2$. Setting $M_A = 0$ and substituting, the length L of the horizontal pipe that would cause the moment at the pipe stem to vanish is determined to be

$$0 = r_1 W - r_2 \dot{m} V_2 \quad \rightarrow \quad 0 = (L/2)Lw - r_2 \dot{m} V_2$$

or

$$L = \sqrt{\frac{2r_2 \dot{m} V_2}{w}} = \sqrt{\frac{2(2\text{ m})(23.56\text{ kg/s})(3\text{ m/s})}{117.7\text{ N/m}}\left(\frac{\text{N}}{\text{kg}\cdot\text{m/s}^2}\right)} = \mathbf{1.55\ m}$$

Discussion Note that the pipe weight and the momentum of the exit stream cause opposing moments at point A. This example shows the importance of accounting for the moments of momentums of flow streams when performing a dynamic analysis and evaluating the stresses in pipe materials at critical cross sections.