Today, we will:

- Begin our discussion of Chapters 9 and 15 – **Differential Analysis of Fluid Flow**

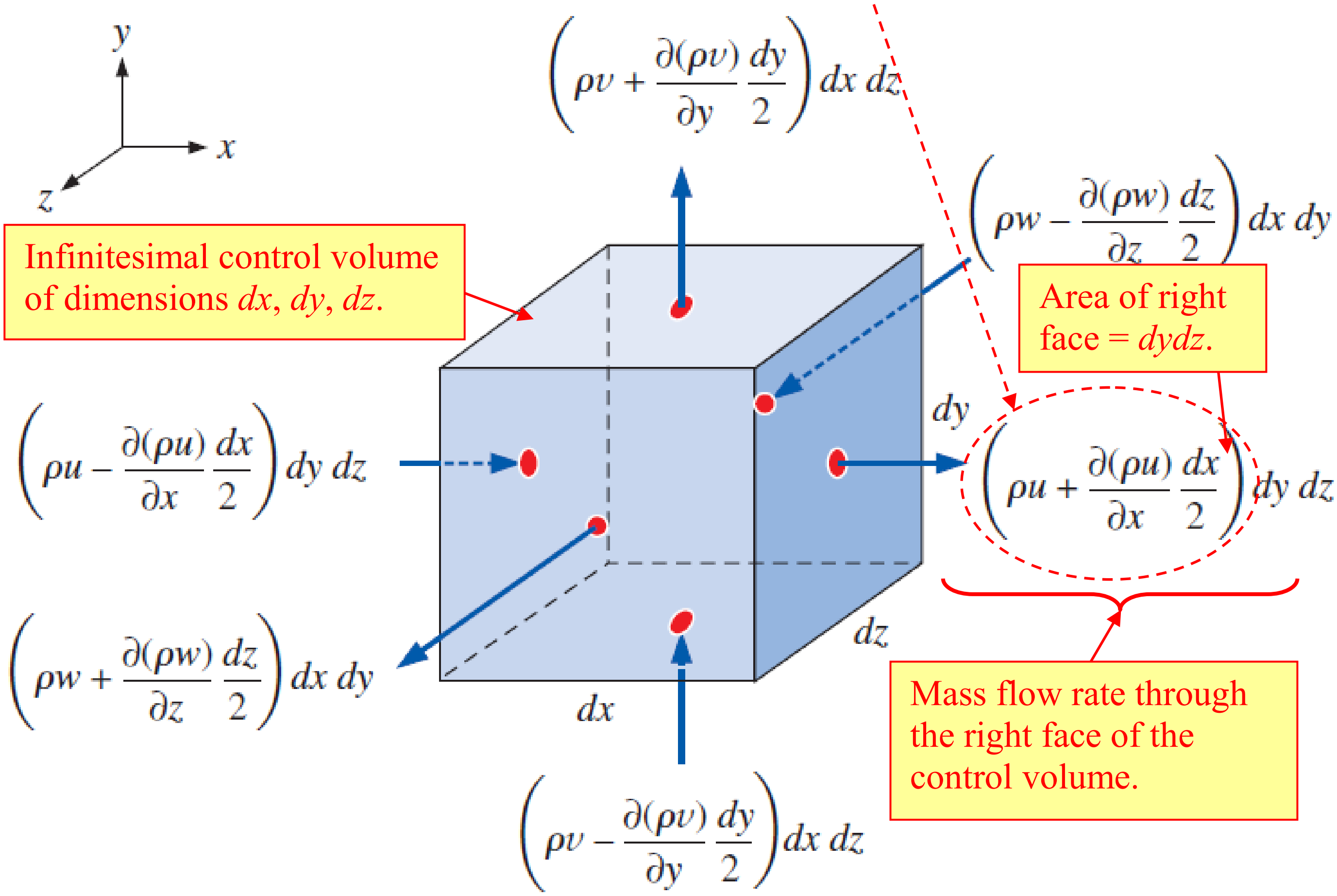
Derivation of the Continuity Equation
(Section 9-2, Çengel and Cimbala)

We summarize the second derivation in the text – the one that uses a ***differential control volume***. First, we approximate the mass flow rate into or out of each of the six surfaces of the control volume, using ***Taylor series expansions*** around the center point, where the velocity components and density are u, v, w, and ρ. For example, at the right face,

Ignore terms higher than order dx.

$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \cdots \qquad \text{(9–6)}$$

The mass flow rate through each face is equal to ρ times the normal component of velocity through the face times the area of the face. We show the mass flow rate through all six faces in the diagram below (Figure 9-5 in the text):



Next, we add up all the mass flow rates through all six faces of the control volume in order to generate the general (unsteady, incompressible) ***continuity equation***:

Net mass flow rate into CV: ← all the *positive* mass flow rates (into CV)

$$\sum_{\text{in}} \dot{m} \cong \underbrace{\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{left face}} + \underbrace{\left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{bottom face}} + \underbrace{\left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{rear face}}$$

Net mass flow rate out of CV: ← all the *negative* mass flow rates (out of CV)

$$\sum_{\text{out}} \dot{m} \cong \underbrace{\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{right face}} + \underbrace{\left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{top face}} + \underbrace{\left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{front face}}$$

We plug these into the integral conservation of mass equation for our control volume:

$$\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, dV = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m} \qquad \textbf{(9–2)}$$

This term is approximated at the *center* of the tiny control volume, i.e.,

$$\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, dV \cong \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

The conservation of mass equation (Eq. 9-2) thus becomes

$$\frac{\partial \rho}{\partial t}\, dx\, dy\, dz = -\frac{\partial(\rho u)}{\partial x}\, dx\, dy\, dz - \frac{\partial(\rho v)}{\partial y}\, dx\, dy\, dz - \frac{\partial(\rho w)}{\partial z}\, dx\, dy\, dz$$

Dividing through by the volume of the control volume, *dxdydz*, yields

Continuity equation in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \textbf{(9–8)}$$

Finally, we apply the definition of the ***divergence*** of a vector, i.e.,

$$\vec{\nabla}\cdot\vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} \quad \text{where} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \quad \text{and} \quad \vec{G} = \left(G_x, G_y, G_z\right)$$

Letting $\vec{G} = \rho\vec{V}$ in the above equation, where $\vec{V} = (u, v, w)$, Eq. 9-8 is re-written as

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \qquad \textbf{(9–5)}$$

3. Examples

Example: Continuity equation

Given: A velocity field is given by

$$u = a\left(x^2 y + y^2\right)$$

$$v = by^2 x$$

$$w = c$$

To do: Under what conditions is this a valid steady, incompressible velocity field?

Solution:

To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Example: Continuity equation

Given: A velocity field is given by

$$u = 3x + 4y$$

$$v = by + 2x^2$$

$$w = 0$$

To do: Calculate *b* such that this a valid steady, incompressible velocity field.

Solution:

To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Example: Continuity equation

Given: A velocity field is given by

$$u = ax + b$$
$$v = \text{ unknown}$$
$$w = 0$$

To do: Derive an expression for v so that this a valid steady, incompressible velocity field.

Solution:

To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{\partial u}{\partial x} \quad + \quad \frac{\partial v}{\partial y} \quad + \quad \frac{\partial w}{\partial z} \quad = 0$$

Example: Continuity equation

Given: A flow field is 2-D in the r-θ plane, and its velocity field is given by

$$u_r = \text{ unknown}$$
$$u_\theta = c\theta$$
$$u_z = 0$$

To do: Derive an expression for u_r so that this a valid steady, incompressible velocity field.

Solution:

To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right) \quad + \quad \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \quad + \quad \frac{\partial u_z}{\partial z} \quad = 0$$

Example: Continuity equation

Given: A flow field is 2-D in the r-θ plane, and its velocity field is given by

$$u_r = -\frac{3}{r} + 2$$

$$u_\theta = 2r + a\theta$$

$$u_z = 0$$

To do: Calculate a such that this a valid steady, incompressible velocity field.

Solution:

To be a vaild steady, incompressible velocity field, it must satisfy continuity!

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) \quad + \quad \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \quad + \quad \frac{\partial u_z}{\partial z} \quad = 0$$